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Curriculum Guideline

Mathematics

Intermediate and Senior Divisions

1985

Part Three

Grades 7 and 8

Grades 9 and 10, Advanced Level

Grades 11 and 12, Advanced Level

Ontario Academic Courses

Curriculum Guideline

Mathematics Intermediate and Senior Divisions

1985

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- Grades 7 and 8
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- Grades 11 and 12, Advanced Level
- Ontario Academic Courses

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Preface

This guideline outlines the expectations of the Ministry of Education for the development of mathematics courses in the Intermediate and Senior Divisions. It includes course outlines for the Ontario Academic Courses in mathematics.

This document supersedes the following guidelines:

- *Mathematics, Curriculum Guideline for the Intermediate Division, 1980*
- *Mathematics, Senior Division, 1972*

A school need not offer all of the courses described in this guideline. However, in the context of such factors as its student enrolment, staff size, and total academic program, a school should ensure that the range of mathematics courses that is available accommodates the specific needs, interests, and mathematical abilities of all of its students.

In reference to the policy statements contained in this document:

- *shall* indicates that which must be implemented;
- *should* indicates that which is strongly recommended;
- *may* indicates that which is optional.

The Mathematics Program: Policy and Directions

Introduction

The study of mathematics is generally viewed as essential in preparing students to become informed citizens in our society. In some form, mathematics is used by virtually everyone, either at work, at home, or in the marketplace.

Mathematics provides a means of communication that is powerful, concise, and usually unambiguous. Through mathematics it is possible to represent complex situations, explain events, and predict outcomes. Mathematics is being used increasingly to create models for analysis and problem solving in a variety of fields, such as science, economics, medicine, art, music, and many of the social sciences.

Mathematics provides a way of thinking that involves the study of patterns, the creation of abstract systems, and the use of logical arguments. The appreciation of the order and structure of a mathematical proof or a mathematical system can be compared to enjoyment of the arts. Just as works of art are interpreted in a variety of ways by different individuals, so the appreciation of much of mathematics is, ultimately, a matter of personal interpretation.

The study of mathematics should be an integral part of the education of all students in the Intermediate Division and of most students in the Senior Division. Through such study students should develop an appreciation of the contributions of mathematics to the development of civilization, of its potential, of its patterns of thought, and of its place in their preparation for prospective careers.

The aims for mathematics courses developed from this guideline are derived from the goals of education established for Ontario.

The Goals of Education

The Ministry of Education in Ontario strives to provide in the schools of the province equal opportunity for all. In its contribution to programs, personnel, facilities, and finances, the ministry has the overall purpose of helping individual learners to achieve their potential in physical, intellectual, emotional, social, cultural, and moral development. The goals of education, therefore, consist of *helping each student to*:

1. **develop a responsiveness to the dynamic processes of learning**
   Processes of learning include observing, sensing, inquiring, creating, analysing, synthesizing, evaluating, and communicating. The dynamic aspect of these processes derives from their source in many instinctive human activities, their application to real-life experiences, and their systematic interrelation within the curriculum.
2. **develop resourcefulness, adaptability, and creativity in learning and living**
   These attributes apply to modes of study and inquiry, to the management of personal affairs such as career plans and leisure activities, and to the ability to deal effectively with challenge and change.
3. **acquire the basic knowledge and skills needed to comprehend and express ideas through words, numbers, and other symbols**
   Such knowledge and skills will assist the learner in applying rational and intuitive processes to the identification and solution of problems by:
   a) using language aptly as a means of communication and an instrument of thought;
   b) reading, listening, and viewing with comprehension and insight;
   c) understanding and using mathematical operations and concepts.
4. **develop physical fitness and good health**
   Factors that contribute to fitness and good health include regular physical activity, an understanding of human biology and nutrition, the avoidance of health hazards, and concern for personal well-being.
5. **gain satisfaction from participating and from sharing in the participation of others in various forms of artistic expression**
   Artistic expression involves the clarification and restructuring of personal perception and experience. It is found in the visual arts, music, drama, and literature, as well as in other areas of the curriculum where both the expressive and receptive capabilities of the learner are being developed.
6. **develop a feeling of self-worth**
   Self-worth is affected by internal and external influences. Internally it is fostered by realistic self-appraisal, confidence and conviction in the pursuit of excellence, self-discipline, and the satisfaction of achievement. Externally it is reinforced by encouragement, respect, and supportive evaluation.

7. **develop an understanding of the role of the individual within the family and the role of the family within society**
   Within the family the individual shares responsibility, develops supportive relationships, and acquires values. Within society the family contributes to the stability and quality of a democratic way of life.

8. **acquire skills that contribute to self-reliance in solving practical problems in everyday life**
   These skills relate to the skilful management of personal resources, effective participation in legal and civic transactions, the art of parenthood, responsible consumerism, the appropriate use of community agencies and services, the application of accident-prevention techniques, and a practical understanding of the basic technology of home maintenance.

9. **develop a sense of personal responsibility in society at the local, national, and international levels**
   Awareness of personal responsibility in society grows out of knowledge and understanding of one's community, one's country, and the rest of the world. It is based on an understanding of social order, a respect for the law and the rights of others, and a concern for the quality of life at home and abroad.

10. **develop esteem for the customs, cultures, and beliefs of a wide variety of societal groups**
    This goal is related to social concord and individual enrichment. In Canada it includes regard for:

    a) the Native peoples;
    b) the English and French founding peoples;
    c) multiculturalism;
    d) national identity and unity.

11. **acquire skills and attitudes that will lead to satisfaction and productivity in the world of work**
    In addition to the appropriate academic, technical, and interpersonal skills, this goal relates to good work habits, flexibility, initiative, leadership, the ability to cope with stress, and regard for the dignity of work.

12. **develop respect for the environment and a commitment to the wise use of resources**
    This goal relates to a knowledgeable concern for the quality of the environment, the careful use of natural resources, and the humane treatment of living things.

13. **develop values related to personal, ethical, or religious beliefs and to the common welfare of society**
    Moral development in the school depends in part on a consideration of ethical principles and religious beliefs, a respect for the ideals held by others, and the identification of personal and societal values.

The preceding goals are not arranged in any hierarchical order, nor are they discrete categories from which a checklist should be made. The integrated nature of learning and the complex pattern of human development preclude such a sequential or fragmented approach. The translation of the goals into curriculum objectives, however, will undoubtedly result in sequences of learning appropriate to the particular levels and stages of development of the students for whom programs are being planned.

While school programs are founded on all the above goals, the planning, development, and teaching of mathematics courses will relate particularly to goals 1, 2, 3, 6, 8, 11, and 12.

Authorized Courses and Credits

The accompanying chart outlines the authorized courses in mathematics and their sequence. Each block on the chart for a secondary school course represents an authorized single-credit course. Courses that are worth a fraction of a credit may be designed by taking subsets of the objectives of an authorized course; no course, however, may be assigned less than one-quarter of a credit. The Ontario Academic Courses in mathematics are to be offered as single-credit courses only.

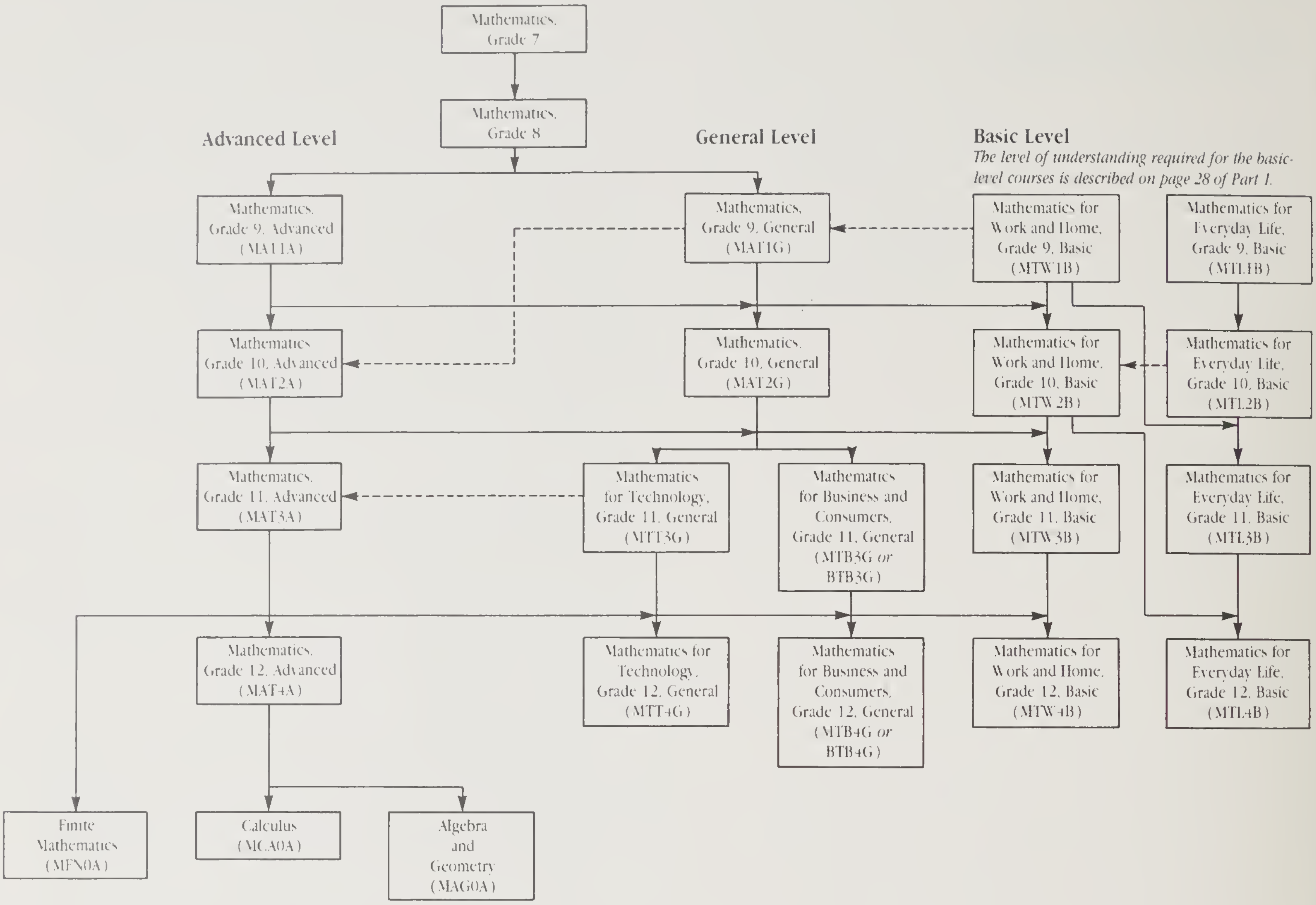
The solid arrows on the chart identify prerequisite courses. Alternate prerequisites for high-achieving students are indicated by broken arrows. These prerequisites are stated in each of the course outlines. For high-achieving, motivated students who wish to specialize in mathematics, Mathematics, Grade 11, Advanced Level, may be considered as a corequisite of Mathematics, Grade 12, Advanced Level, rather than as a prerequisite. Transition courses, which are assigned fractional credits, may be designed to provide for students who do not have a required prerequisite (see *OSIS*, section 5.4). In cases where individual students or parents request exemption from a prerequisite course, the principal of the school will rule on the request. Schools must provide clear information on such prerequisites to students and parents. No courses shall be identified as prerequisites other than those so identified in this guideline.

The successful achievement of a mathematics course that has been repeated at a different level of difficulty may be granted a full credit, fractional credit, or no credit depending on the judgement of a school's principal and staff as to the amount of new work involved (see *OSIS*, section 5.4).

The courses designed for Grades 9-12/OACs should be offered at the three levels of difficulty as outlined as follows in section 4.6 of *Ontario Schools, Intermediate and Senior Divisions (Grades 7-12/OACs): Program and Diploma Requirements, 1984 (OSIS)*:

> **Basic Level**
>
> Basic-level courses are designed to focus on the development of personal skills, social understanding, self-confidence, and preparation for the world of work. The academic work and related skills should be perceived by the student as being personally useful. Such courses will assist students to prepare for a successful, independent home and working life, to manage personal financial resources, to communicate effectively, and to develop attitudes that foster respect for the environment, good health and fitness, and a positive approach towards work and leisure. These courses should serve the needs of the student who may not participate in post-secondary education and provide a good preparation for direct entry into employment.

Mathematics Program and Prerequisites


Mathematics, Grade 7
Mathematics, Grade 8
Advanced Level
General Level
Basic Level
The level of understanding required for the basic-level courses is described on page 28 of Part 1.
Mathematics, Grade 9, Advanced (MAT1A)
Mathematics, Grade 9, General (MAT1G)
Mathematics for Work and Home, Grade 9, Basic (MTW1B)
Mathematics for Everyday Life, Grade 9, Basic (MTL1B)
Mathematics, Grade 10, Advanced (MAT2A)
Mathematics, Grade 10, General (MAT2G)
Mathematics for Work and Home, Grade 10, Basic (MTW2B)
Mathematics for Everyday Life, Grade 10, Basic (MTL2B)
Mathematics, Grade 11, Advanced (MAT3A)
Mathematics for Technology, Grade 11, General (MTT3G)
Mathematics for Business and Consumers, Grade 11, General (MTB3G or BTB3G)
Mathematics for Work and Home, Grade 11, Basic (MTW3B)
Mathematics for Everyday Life, Grade 11, Basic (MTL3B)
Mathematics, Grade 12, Advanced (MAT4A)
Mathematics for Technology, Grade 12, General (MTT4G)
Mathematics for Business and Consumers, Grade 12, General (MTB4G or BTB4G)
Mathematics for Work and Home, Grade 12, Basic (MTW4B)
Mathematics for Everyday Life, Grade 12, Basic (MTL4B)
Finite Mathematics (MFN0A)
Calculus (MCA0A)
Algebra and Geometry (MAG0A)

Note: A high-achieving, motivated student may treat Mathematics, Grade 11, Advanced, as a corequisite of Mathematics, Grade 12, Advanced, rather than as a prerequisite.

Legend: ——→ prerequisite courses
- - - - → alternate prerequisites (with high achievement)

For statement of prerequisites refer to the individual course outlines.

General Level

General-level courses should be considered as appropriate preparation for employment, careers, or further education in certain programs in the colleges of applied arts and technology and other non-degree-granting post-secondary educational institutions.

General-level courses will be designed in terms of a mix of the following aims:

- to prepare students to read, write, listen, and speak with confidence and clarity;
- to prepare students to participate as active and compassionate citizens in a democratic society;
- to prepare students to develop the attitudes and skills that will permit them to enter directly into employment on graduation or into certain programs at the colleges of applied arts and technology;
- to prepare students to develop the habit of learning those new, personal skills (problem-solving, domestic, consumer, recreational) that they may need throughout their lives;
- to acquaint students with the language, assumptions, issues, and career opportunities of various subject or social disciplines;
- to stimulate students to continue to develop and increase their awareness, appreciation, and enjoyment of and skill in the arts.

The priority among these aims will vary with the focus of any given course, but it is expected that the first four aims will normally be integral to any general-level course . . . .

Advanced Level

Advanced-level courses should focus on the development of academic skills and prepare students for entry to university or to certain programs of the colleges of applied arts and technology. Such courses should be designed to assist students to understand the theoretical principles, practical applications, and substantive content of a subject. All of the aims that have been described for general-level and basic-level courses can be incorporated into advanced-level courses. Communications skills, citizenship preparation, aesthetic awareness, and personal skills of various kinds are essential components of most advanced-level courses, but the focus of these courses must be on academic preparation.

The teaching strategies and evaluation practices shall reflect the different objectives outlined for each course from Grade 7 to 12/OACs. The depth to which each objective is emphasized is left to the discretion of the teacher. A suggested time frame is provided for each of the Grade 7 and 8 courses and for each course at the general or advanced level of difficulty as an indication of the emphasis that might be allotted to the objectives of each section. A time frame is not suggested for basic-level courses, since those courses should provide maximum flexibility for responding to individual student needs.

The outline for the design of the Ontario Academic Courses contains a suggested teaching sequence. However, there is no teaching sequence implied by the guideline for each of the other courses for Grades 7-12. Courses should be designed to integrate appropriate objectives from different sections of each course outline. The development of Grade 8 and 9 courses should involve communication between secondary school teachers and the teachers of related feeder schools.

Course design based on this guideline shall include a consideration of the process components outlined in this introductory part of the guideline, as well as the specific content objectives outlined for each course. Any of the courses authorized by this guideline may be adapted for students who have been identified as exceptional and who require special educational assistance.

The course names and codes identified in this guideline shall be used when reports are made to the Ministry of Education and on the Ontario Student Transcript.

Roles in Program Development

While the Ministry of Education establishes goals, policies, and specific expectations through its curriculum guidelines, it is the responsibility of each school board to:

- ensure that outlines of the courses of study consistent with this guideline are available in all schools offering such courses;
- plan and co-ordinate local curriculum development;
- assist those staff members who are responsible for developing courses of study and materials based on this guideline.

Ongoing staff development is an essential component of school board planning and co-ordination. The curriculum implemented in schools should be reviewed periodically to determine discrepancies between the practice and the philosophical intent of a guideline (see "Evaluation of Program Effectiveness", page 24).

In-service programs should be planned to assist teachers in implementing both the process and content objectives. Where changes in attitude and/or entrenched practice are required, such planning should incorporate long-term in-service programs. Boards should seek the co-operation of personnel from regional offices, faculties of education, universities, colleges, and professional and subject organizations in planning and delivering in-service programs.

Faculties of education are expected to use this guideline in the preparation and delivery of preservice and in-service courses for teachers.

This guideline provides the direction that educators require to plan and teach specific courses. Once the implementation of this guideline is initiated in schools, it is the responsibility of board supervisory officers and principals to ensure that the expectations of the guideline are being met.

In developing the mathematics program for a school, teachers should create an overview of the various courses. This should be constantly updated so that all concerned will be aware of the topics that have been introduced from grade to grade. This is essential for courses at the basic level of difficulty.

Outlines of courses of study developed from this guideline shall be on file in schools, and their effective use in classroom programs should be monitored through regular visitations and evaluations by principals or supervisory officers.

In schools where numbers preclude the organization of full classes, the courses thus affected may be offered by splitting classes on a grade basis (Grade 9, Basic Level, combined with Grade 10, Basic Level) or on a course basis (Mathematics for Business and Consumers, Grade 11, combined with Mathematics for Technology, Grade 11). In such cases students shall be clearly identified as being in a specific course, and the objectives and assessment strategies for the different groups of students shall be distinct. (See OSIS, sections 4.6 and 5.5.)

It is essential that meaningful curriculum review and renewal take place on an ongoing basis. The effectiveness of this curriculum guideline depends on communication, co-operation, and co-ordination among ministry personnel, school board officials, and teachers. This co-operative effort is essential to assure the continual improvement of learning experiences for students.

Cross-Curricular Components

Mathematics Anxiety

The avoidance of mathematics is often associated with a mathematics anxiety, which can be alleviated by concerned teachers. A supportive classroom environment, which can decrease anxiety, should involve aspects such as the following:

- the provision of many opportunities for students to experience success and positive reinforcement;
- patience, receptivity, and understanding;
- the avoidance of unnecessary tension and pressure in the classroom;
- a matching of the reading level of resources to that of students;
- the use of a variety of assessment techniques (e.g., frequent short tests, take-home tests, projects);
- provisions for retesting, if necessary;
- the acceptance of alternative solutions;
- the use of diagnostic interviews to probe beyond a superficial identification of learning problems associated with mathematics;
- the relating of mathematical concepts and generalizations to applications familiar to the learner.

Individual Differences

Each student enrolled in Intermediate or Senior Division mathematics courses has his/her own personal way of perceiving the world. Individuals vary in the ways they receive, process, recall, apply, and transmit information. Each student will have a preferred learning style. Some will benefit more from oral instruction, others from the written word; some from manipulating concrete materials, others from working with abstract ideas; some from self-directed activity, others from the structured presentation of ideas; some from individual effort, others from group interaction. Although some individual differences will be addressed through the student's selection of a course at an appropriate level of difficulty, teachers should be prepared to identify the learning styles, needs, and abilities of students in any class and to make the necessary adjustments in the design, development, and presentation of the course to accommodate them.

For example, in order to help a student who understands mathematical ideas when they are presented through manipulative materials or when they are related to familiar events but has difficulty when they are presented abstractly, the teacher might:

- associate mathematical concepts with familiar objects or applications;
- use inductive rather than deductive reasoning;
- provide step-by-step instructions when lengthy procedures are involved;
- encourage a check of conclusions through the use of familiar ideas such as numbers or simple geometric shapes;
- provide immediate reinforcement of new concepts and procedures through their application to familiar situations.

In order to help a student who understands visual presentations and spatial relationships but has difficulty with auditory discrimination, memory, and symbolization, the teacher might:

- restrict distracting noise;
- keep verbal explanations concise and relevant to the task;
- exaggerate inflection and emphasis on key concepts or terms;
- use gestures and visual demonstrations in conjunction with oral instruction;
- present appropriate vocabulary in conjunction with experience to assist associative memory;
- provide sufficient time for the student to formulate his/her thought and to plan an oral response.

In order to help a student who understands and retains oral presentations but has difficulty with visual discrimination, memory, and symbolization, the teacher might:

- reduce visual distractions;
- focus the student's attention on specific aspects of diagrams or algebraic expressions through oral clues, gestures, and the restricted use of colour;
- provide sufficient time between a visual presentation and an oral response so that the student can interpret what he/she has seen, formulate his/her thought, and plan a response;
- encourage the student to verbalize while doing assignments.

The following are some strategies that can be used to identify individual differences and to accommodate them:

- observation, interviews, and tests to provide a basis for the diagnosis of individual strengths and weaknesses;
- the assigning of individual projects that span a wide range of interests and skills;
- the organization of the class into several homogeneous groups, each assigned different activities appropriate to the learning styles, needs, or abilities of the members of the group;
- the organization of the class into several heterogeneous groups that span a wide range of interests and skills;
- the pairing of students so that they can work together on assigned tasks;
- the designing of seatwork and homework assignments that are differentiated for individuals or groups within the class;
- the designing of seatwork and homework assignments to provide for individual choice;
- the use of appropriate computer software;
- the provision of enrichment activities for individuals;
- the assessment of individuals on differing aspects of the course (e.g., core only, core plus optional, core with emphasis on knowledge and recall, core with emphasis on application and problem solving);
- the evaluation of individuals to take into account needs, abilities, and attitudes, as well as achievement;
- the use of reporting that recognizes individual differences.

Throughout the Intermediate and Senior Divisions each student should be provided with increasing opportunities for self-direction and self-discipline.

The Adult Perspective

This guideline may serve as a basis for courses for adult learners in regular school programs in mathematics. Such classes may be composed exclusively of adults, as in continuing education, or of adults integrated with school-age students.

Course planning for adult learners should include a consideration of adult-learning principles, including the following:

- Adult learners often bring a rich store of practical experience with them. Classroom activities can be designed to elicit and build on this base.
- Many adults learn best in groups and welcome the support of their fellow learners. Activities that foster social interaction will result in more effective learning. Through such activities learners become a resource for one another.
- Many adults learn effectively through active participation. Strategies that balance experiential activities with theoretical and abstract presentations may enhance their learning.
- Many adults tend to focus on the present and to be problem-oriented. The efforts of the teacher to discover and build on their expressed learning needs will generally enrich the learning situation.
- Adults exhibit a far greater range of learning styles than do school-age students. Opportunities for independent study and for learning projects may be used to provide for individual differences.

Diagnostic testing, interviews, and planned remedial experiences will help adult learners to adjust to current mathematics programs. Diagnosis should concentrate on those concepts and skills that are deemed essential for attaining the objectives of the course being taken, rather than on all the objectives of prerequisite courses. The adult learner should be assisted to make the bridge between previously learned concepts and current mathematical vocabulary and symbols.

Adult-learning theory suggests that adults tend to be highly self-directed, responsible, and mature. The teacher should decide the extent to which the principles described above may be implemented in a given situation with the learners in the class.

Exceptional Students

Some exceptional students can be accommodated in a regular mathematics classroom with only minor changes in objectives or through an emphasis on certain teaching techniques. Other exceptional students will require more extensive modification of the program in terms of pacing, breadth or depth of content, methods of assessment, or teaching strategies.

Five broad areas of exceptionality – behavioural, communicational, intellectual, physical, and multiple – delineate the range of differences for which provisions must be made. The Identification, Placement, and Review Committee of a board identifies students as exceptional and states the identification it has made of the learning needs of each student. To meet these needs, courses designed from this guideline should provide students with learning experiences that correspond in content, process, product, and evaluation to the students' needs, abilities, interests, and aspirations. Alterations should be made to the program to accommodate the unique needs of the exceptional student in respect to the rate of progress and level of difficulty, but, at the same time, the integrity of this guideline should be maintained. The assistance of additional professional staff and the use of specialized equipment or facilities may be required.

A special education program planned for an exceptional student must be based on and modified in accordance with the results of continuous assessment and evaluation. The use of a variety of assessment techniques should ensure a comprehensive evaluation of each student's progress. Subsequent to evaluation, further alterations to a student's program may be required in terms of teaching/learning strategies or a more suitable combination of compulsory and optional credits.

Exceptional students may select courses from any of the three levels of difficulty. However, teachers and principals should ensure that these students have their needs considered and that any necessary modifications are incorporated within the selected courses. Reference should be made to appropriate support documents for exceptional students for suggestions related to planning, teaching, and evaluating courses.

A student who is intellectually handicapped might be helped by a teacher who:

- finds the level at which success is achieved and proceeds from there at a pace that sustains the student's success;
- provides the student with individual help and attention;
- uses objects that the student can manipulate and to which he/she can relate (e.g., money, a ruler, chips, or dice);
- proceeds in short, single, systematic steps, repeating these if necessary;
- develops the student's understanding, accuracy, and speed in that order;
- provides the student with motivation;
- establishes individual standards of quality;
- develops abstractions, where necessary, using manipulative materials and imagery;
- allows the students to refer to resource materials when performing mathematical tasks.

A mathematically gifted student might be helped by a teacher who:

- groups the student with other gifted students so that they all receive the encouragement and stimulation of peers;
- provides problems that encourage students to develop generalizations;
- encourages the formulation of mathematical relationships;
- encourages the development of a variety of solutions to problems;
- provides the student with the opportunities and the incentive to pursue chosen mathematical topics independently and in depth;
- encourages the creation of new problems based on one already considered;
- arranges for the student's participation in mathematics clubs, leagues, and fairs;
- arranges student contacts with mentors;
- provides the student with opportunities for preparing, presenting, and participating in seminars.

Mathematically gifted students will normally solve problems swiftly and with insight, sometimes seeming to omit some steps in logic that less gifted students require. They will tend to generalize broadly and rapidly; to show flexibility in thinking; to be creative in their solutions; to be persistent when doing mathematics; and to perceive many situations in mathematical terms. While mathematically gifted students will usually be achieving well in mathematics, they may not be the highest achievers. Conversely, many high achievers may not be mathematically gifted.

Careers and Mathematics

There are few occupations in our society that do not use mathematics in some form, and few, if any, consumers who do not employ some mathematical ideas in their purchasing decisions, record keeping, or leisure activities. Mathematics teachers shall discuss with students how mathematical knowledge relates to and affects entry into a variety of occupations. They should assist students in career planning through counselling that relates students' interest, aptitude, and achievement in mathematics to career goals. They should make specific reference to applications related to occupations that are important to their students.

The following lists indicate the level of mathematics associated with a number of occupations (at the time of development of this guideline):

- ***secondary school mathematics beyond Grade 10 desirable:*** accounting clerk, animal attendant, bank teller, building tradesperson, fashion designer, geographer, interior designer, lawyer, mechanic, salesperson, stenographer
- ***college mathematics desirable:*** agricultural technician, computer operator, insurance underwriter, medical technologist, nurse
- ***college mathematics essential:*** business administrator, computer programmer, machinist, optician, engineering technician, engineering technologist

- ***some university mathematics useful:*** accountant, architect, dentist, elementary school teacher, food scientist, psychologist, social scientist
- ***university mathematics essential:*** actuary, applied mathematician, astronomer, chemist, mathematics teacher, meteorologist, physicist, professional engineer, statistician, systems analyst

For current information on the level of mathematics desirable or required for these or other specific occupations, mathematics teachers should seek the assistance of guidance departments and refer students to the Student Guidance Information Service (SGIS) and the library resource centre.

Statistical data indicate that the participation of female students in mathematics courses drops significantly after Grade 11. This tends to limit their chances of entry into technical and professional fields. Technology programs require Grade 12 mathematics as a prerequisite, and mathematics is increasingly used in the social, behavioural, managerial, and life sciences, as well as in physics, chemistry, and engineering.

The following are some of the factors that contribute to an individual's participation in mathematics courses:

- the student's perception of the need for and usefulness of mathematics in careers;
- the student's attitude towards mathematics;
- the encouragement of important persons such as teachers, parents, and counsellors.

Teachers should encourage both male and female students to continue their study of mathematics by consciously:

- developing in students an awareness of the usefulness of mathematics in various occupations and everyday life;
- encouraging the attitude that mathematics is open to both sexes equally;
- building student confidence through a supportive environment.

The ability of secondary school graduates to take advantage of opportunities for career advancement or change can depend on the number of courses in mathematics that they have completed. In order to ensure that female graduates will not be at a disadvantage because they have avoided optional mathematics courses, mathematics teachers should motivate and prompt all female students to continue with mathematics at a level commensurate with their mathematical abilities.

Values Education

Values education is an integral part of the school experience. In the mathematics classroom it is important that the teacher provide students with regular opportunities to reflect on the values and issues that arise from the subject matter and from the interaction of the students and the teacher. These opportunities will help students to clarify and carefully examine values within the social context, as well as to learn reflective skills that will assist them with the value dilemmas they will meet in their own lives.

Some applications related to areas such as economics (e.g., How much of earned income should be spent on a car?) and the environment (e.g., What are the implications of the exponential growth of pollutants?) provide direct opportunities to address values issues. However, the mathematics classroom should focus on the nurturing of some of the values that are the foundation of Ontario school programs. These include respect for self and others, tolerance, patience, courtesy, and co-operation. Each of these values can be demonstrated and encouraged in the context of the varying perceptions of mathematics and the variety of solutions that challenging problems will generate. In addition, the mathematics teacher can encourage the use of problem-solving strategies that may be helpful in resolving value conflicts.

The consideration of values and issues takes place within a classroom environment that maximizes concern for justice, respect, and caring. Students should be challenged to consider the ethical implications of decisions and to become increasingly aware of both rights and responsibilities.

Process Components in the Mathematics Program

The development of the following process components from grade to grade shall be a consideration in local curriculum development. These components shall, in varying degrees, be part of all courses designed from this guideline. Some guidance related to specific courses and sections of courses is given within the lists of content objectives.

Language and Mathematics

Language (listening, speaking, reading, and writing) plays a fundamental part in the teaching and learning of mathematics. Not only is language a means of communication, but it is also part of a thinking process through which ideas are combined, relationships found, and problems posed and solved. Teachers should appreciate the distinction between language used to develop an understanding of mathematics and the language of mathematics. They should recognize this distinction in planning and providing instruction.

Language used to develop an understanding of mathematics. The role of language in both communication and understanding can be strengthened through the use of questions that probe for explanations rather than stimulate recognition or recall. Students should be consistently asked "Why?" and helped to use language that expresses *their* answer to such a question and conveys *their* meaning to others. Opportunities should be provided for students to engage in writing activities in which they explore their own perceptions of mathematical concepts and report on their attempts to apply mathematics to problem solving. Students should also be provided with opportunities to engage in speaking and listening activities in which they test their thinking and reasoning against those of their peers. It is through these types of activities that students modify, verify, and consolidate their learning.

Where appropriate, students should be given opportunities to learn mathematics independently through reading, in full recognition of the problems inherent in reading mathematics. Mathematics textbooks will not usually provide good independent-reading assignments. Such assignments are better based, initially, on books that have been written for the general public. Students in Senior Division courses preparing for the study of mathematics at university must have experiences in learning independently from textbooks.

Mathematics as a language. The precision of mathematical vocabulary and the conciseness of mathematical symbolism is often difficult for the learner. Each unit in a course should be analysed to identify such vocabulary and symbolic content. Questions such as the following are useful in this regard: How is this vocabulary or notation related to everyday usage? How does it conflict with everyday usage? How is it related to previously introduced mathematical terms? The answers to these questions will suggest ways in which students can learn and consolidate the special mathematical meanings of the terms and symbols. It is important to recognize the role of experience in developing concepts. Definitions should usually be developed as summarizing statements of ideas that have been understood, not as the initial introduction of a term, symbol, or concept.

The vocabulary of mathematics can be placed into three main categories:

- words and symbols that have both an everyday meaning and a special meaning in mathematics (e.g., function, negative, similar);
- words and symbols that have a unique meaning in mathematics (e.g., decimal, $\sqrt{\ }$, π, logarithm);
- words and symbols that are unique to mathematics, but that have multiple interpretations within mathematics (e.g., -1 as used in $x-1$ or a^{-1} or $f^{-1}(x)$, and root as related to numbers or equations).

For clarity and understanding, communication using the language of mathematics must conform to the rules of that language. The misuse or careless use of symbols in a statement can lead to misinterpretation even by the person who writes the statement. Thus, students should be continually made aware of good form, and mathematics instruction should include means of assisting students to overcome or to avoid specific difficulties associated with the language of mathematics.

Experiential Approaches

Manipulative materials and pictorial representations play a crucial role in students' mastering of mathematical concepts. Throughout the Intermediate Division and for many students in the Senior Division, both the review of previously encountered concepts and the development of new ones should take place through approaches that relate abstractions to manipulative materials or to real-life situations that may be experienced or simulated in the classroom. The selection of a specific experience should be based, in part, on its familiarity to the students and its usefulness in assisting recall through mental images.

The experiential approach is important to students in the Intermediate Division, especially for their eventual understanding of concepts related to fractions, ratio, percent, decimals, algebra, measurement, and geometry. In the Senior Division, the development of students' understanding of trigonometry, vectors, relations, and equations is dependent on their having mental images based on experiences that give meaning to the abstract concepts involved. Where applicable, carefully selected computer software should be used to provide experience for concept development. For some students, writing their own computer programs may contribute to such development.

When an experiential approach is used, its success depends on the teacher's guiding students to recognize the patterns and relationships that lead to the desired mathematical concept or generalization.

Proof

Logical proof involves the process of reasoning from a set of premises through a series of connected inferences to a conclusion. In mathematics, there are five major methods of proof: direct proof, proof by use of the contrapositive, indirect proof (reductio ad absurdum), proof by listing all possible cases, and proof by mathematical induction. There are also two methods of disproof: disproof by contradiction and disproof by counter example.

While some students, even on completion of the Senior Division, will not be able to appreciate all of these methods of proof and disproof, many students, even in the Intermediate Division, can appreciate proof by listing all possible cases, disproof by contradiction, and disproof by counter example. Most students will not engage successfully in direct proof until the Senior Division because of their stage of development. For those who are ready, the growth of their reasoning ability will depend upon continued experiences in using logical proof. In the Intermediate Division the processes of proof should be developed by challenging students to justify statements, to search for all cases, to look for counter examples, and to demonstrate the existence of a number or a geometric figure with certain properties. The teacher should model logical thinking through the consistent use of conditional reasoning (e.g., "if ... then" sentences).

Both teachers and students should "think aloud" in analysing and proving problems. Attention should be drawn to the differences between the problem-solving strategies and the ultimate structured proof. Beginning in the Grade 10 advanced-level mathematics course, attention shall be directed to the role of the building blocks of proof (undefined terms, definition, assumptions) with geometry used as a model for a logical system. The Grade 12 advanced-level course provides for the extension of this to more complex proofs. In the Senior Division and particularly in advanced-level courses, the role of proof as the culmination of problem solving (proving a solution true in general) should be emphasized.

When considering the evaluation of a student's knowledge of proof, the teacher should remember that it is an outgrowth of a mixed set of skills, habits, and attitudes. These include being aware of assumptions, listening to and evaluating arguments, recognizing patterns (including cases where patterns cannot be extended without proof), and being willing and able to think hypothetically. There may be few students, even in the Senior Division, who can be expected to demonstrate all these attributes consistently.

Applications

Everyone uses mathematics, in some form or other, in everyday life. The curriculum based on these guidelines shall be designed to help students to recognize these uses, to develop an understanding of them, and to extend this understanding to new applications. The ultimate goal is to have students transfer their present use of mathematics to unfamiliar situations. Thus, the mathematics program for the Intermediate and Senior Divisions should give students many opportunities to see a direct relationship between their studies and the real problems that they face now or will encounter in the future.

Wherever possible, skills and concepts should be related, from the beginning, to their applications. This approach can assist students to understand mathematical principles and to establish a secure base on which they can build more sophisticated ideas and consider more complex applications.

Applications shall be interspersed throughout the program, rather than developed in isolation. An effort should be made to relate the mathematics program to programs in other subject areas, such as science, technological studies, social sciences, business studies, and the arts. The applications chosen should also relate to the interests of students, the levels of difficulty of a course, and their relevance in the community.

Calculators or computers shall be used in most work based on applications, since such aids enable teachers to extend the breadth and depth of the problems explored.

Mathematical Models

The process of problem solving using mathematics inevitably involves the choice or creation of a mathematical model. As a problem becomes more complex, the selection of an appropriate mathematical model becomes more crucial in the attainment of a solution. In the Intermediate Division the models used may initially involve numerical expressions, tables, graphs, diagrams, and eventually formulas, linear equations, and geometric relations. In the Senior Division the models used may also involve quadratic equations, trigonometry, series, vectors, matrices, and calculus.

Throughout all of the courses work on applications and problem solving should emphasize the aspect of choice in mathematical modelling. Students should come to appreciate the stages involved in working with an application. These stages are:

1. identifying the problem to be solved;
2. modifying and simplifying the problem so that the important features are described with reasonable precision (a real model);
3. translating the ideas of the real model into a mathematical model;
4. deriving some conclusions as a result of a consideration of the mathematical model;
5. testing the conclusions by comparison with the results in the application;
6. if necessary, modifying the real or mathematical model or selecting a new one so that better predictions can be made.

Many of these stages will involve the strategies of problem solving described in the next section.

In the classroom context teachers often introduce a very precisely defined and simplified real model, but they should also be watching for opportunities to involve students in situations or questions that are less precisely defined. Outdoor-education experiences often provide such opportunities (e.g., How high?, How far across?), as do management problems (e.g., How can the cafeteria be redesigned to handle peak traffic?).

Problem Solving

Developing the ability to solve problems is a major goal of mathematics education. Problems are solved by drawing on past experiences – sometimes in a systematic manner, but often in flashes of creativity and intuition. Problem solving is not exclusively the domain of mathematics; it is an integral part of all subjects and of everyday life.

Systematic problem solving involves the following stages:

1. the awareness of a situation in which there is given information and a goal;
2. a) the consideration of possible strategies;
   b) the choice of a strategy;
3. the carrying out of the strategy;
4. the verification of the solution in the situation.

There are two major types of problems:

- A strategy is evident immediately. Difficulties in solution are related to one's ability to carry out the strategy correctly.
- A strategy is not immediately evident. Difficulties in solution are initially related to choosing an appropriate strategy.

A given problem may be of the first type for one individual, but of the second type for another individual.

For most students word problems assigned after a mathematical concept, generalization, or skill has been taught are usually of the first type, requiring only the application of a known algorithm for solution. It is essential that students also have experiences throughout each grade with problems of the second type. Generally, these problems should be solvable by a variety of strategies or by models and techniques that have not been recently taught or practised, or in a combination not previously experienced.

The following procedures should be stressed in connection with the stages of systematic problem solving:

1. Identifying relevant and irrelevant information. This involves reading, understanding, paraphrasing, summarizing, and listing.
2. a) Identifying possible strategies. These include:
      - classifying information (e.g., as insufficient, conflicting, extraneous, redundant);
      - searching for a pattern;
      - drawing a diagram or flow chart;
      - constructing a table;
      - estimating (guessing and checking; improving the guess);
      - choosing mathematical operations and sequencing them;
      - assuming a solution and working backwards;
      - using a formula or writing an equation;
      - solving a simpler problem (part of the problem);
      - accounting for all possibilities;
      - checking for hidden assumptions;
      - making an assumption and drawing a conclusion.

b) Selecting a strategy based on some of the following considerations: familiarity, ease of implementation, efficiency (elegance).

3. Carrying out the strategy as follows:
    - working with care;
    - checking work;
    - presenting ideas clearly;
    - persisting (trying, resting, trying again, trying another strategy).
4. Determining how good the solution is by:
    - verifying it in the problem situation (reasonableness of result);
    - generalizing the solution to similar problems;
    - searching for a better solution.

Throughout the Intermediate and Senior Divisions courses at all levels of difficulty shall include planned experiences based on the second type of problem, which will help students to strengthen the problem-solving skills that have been summarized above. Some of these experiences could be situations or applications that encourage problem posing.

Use of Calculators

The calculator has become an integral part of our way of life. In recognition of this fact, schools should ensure that students become proficient and discerning in the use of calculators. Estimation and mental calculation should be used to anticipate and verify calculator results.

Students should possess some reliable and efficient methods of carrying out calculations without the use of a calculator when a small number of digits is involved and with a calculator when a large number of digits is involved.

Calculators shall be used when the primary purpose of a given activity is the development of problem-solving or other skills in which computation is of secondary importance. In all mathematics courses student facility with appropriate calculator skills should be assessed. Such skills should be taught as necessary.

Computers in Mathematics

As the cost of good educational hardware decreases and the availability of good educational software increases, teachers will be presented with increasing opportunities for enhancing or restructuring mathematics programs. The use of computers should change the quantity and type of interactions in the classroom and encourage co-operative learning. As well, the relative emphasis placed on the various objectives within mathematics courses should be adjusted in the light of the changed perception that computer technology brings.

In the planning and development of courses, consideration should be given to relevant ways of incorporating knowledge, skills, and attitudes related to the computer. Alternative strategies for teaching and learning mathematics using the computer should be explored.

Estimation

It is important to develop in students the ability, either in advance or in retrospect, (a) to determine whether a solution in problem solving is reasonable and (b) to check computation mentally when using a calculator. Estimation is possible only when the context of a problem and the nature of computation are understood. It is a set of skills that is developed over time through practice. The assessment of the acquisition of these skills is best done through informal observation, since formal assessment tends to encourage the student to make an approximation after obtaining an exact answer.

To develop the skills of estimation, students should be asked to:

- state a largest and smallest reasonable answer to a problem before solving the problem. The emphasis should be on providing a basis for judging the solution, not on making a "guess";
- predict whether computation(s) will result in a larger or smaller number than they started with;
- forecast an order of magnitude for the result of computations (such as 10s, 100s, 1000s);
- anticipate the magnitude of the result of a computation by rounding numbers to one significant digit.

Estimation should be encouraged in all activities involving applications, problem solving, and the use of calculators so that students make a habit of checking the reasonableness of their results and the magnitude of their answers.

Accuracy and Precision

In problem solving related to applications, students should be encouraged to consider the implied accuracy of their answers in comparison with the precision of the numbers used to calculate the answer. If the context of a problem does not supply information about the precision of the numbers involved, then approximations made in the process of calculation should assume that the original data were exact. In such cases the answer should be stated with a qualification as to its precision, such as "correct to the nearest cent" or "correct to two decimal places". In most cases measurements are assumed to have a precision of ±0.5 of the last significant place given.

Students studying subjects such as physics or technological studies may encounter statements of precision expressing ranges of tolerances in other forms, such as "10 g ± 1.70 g" or "1.50 cm ± 0.4 percent". Mathematics teachers should co-operate with other teachers in the school so that calculations are dealt with consistently.

The following rules from the *Canadian Metric Practice Guide* provide a basis for approximations in the process of calculation. These should be used, particularly in advanced-level mathematics courses in the Senior Division and in the Mathematics for Technology courses, to help make decisions about approximations. While students in Grades 7 and 8 need not know the rules, when approximations are necessary, the rounding done should be consistent with the rules. Some calculations involving the implied tolerances of numbers could be used to justify the rules.

Rounding of Data.
When a figure is to be rounded to fewer digits than the total number of stated digits, the procedure should be as follows:

a) When the first digit discarded is less than five, the last digit retained should not be changed.
Example: 3.141 326 rounded to 4 digits 3.141.

b) When the first digit discarded is greater than five, or if it is a five followed by at least one digit other than zero, the last figure retained should be increased by one unit.
Examples: 2.213 72 rounded to 4 digits 2.214;
4.168 501 rounded to 4 digits 4.169.

c) When the first digit discarded is five, followed only by zeros, the last digit retained should be increased by one if it is odd, but no change made if it is even.
Examples: 2.35 rounded to 2 digits 2.4;
−6.35 rounded to 2 digits −6.4;
2.45 rounded to 2 digits is 2.4.

Canadian Metric Practice Guide (Rexdale, Ont.: Canadian Standards Association, 1979), page 39

Note: For business and consumer applications, students should be taught the common practice of rounding up (or down) to the nearest cent or dollar regardless of the size of the first digit discarded.

Addition and Subtraction.
When numbers are added or subtracted, the result after the answer is rounded should be as precise as the least precise of the numbers used in the computation.
Example: $0.5392 + 6.11 + 0.284 + 3.018 = 9.9512$
Round to 2 digits after the decimal marker, giving the final result as 9.95.

Multiplication and Division.
When numbers are multiplied or divided, the result after the answer is rounded should contain as many significant digits as are contained in the number with the fewest significant digits that is used in the computation. To have more digits in the answer would be unwarranted and misleading. Example: $453.592 \times 2.68 = 1215.626\,56$
Round to 3 significant digits, to give the final result as 1.22×10^3.
Canadian Metric Practice Guide (Rexdale, Ont.: Canadian Standards Association, 1979), page 39

For further suggestions regarding accuracy and precision, refer to the science guideline for the Intermediate and Senior Divisions.

Evaluation

The goal of evaluation is to improve both student learning and the quality of the program (including materials, teaching, and assessment). The evaluation of student achievement and of program effectiveness shall be an integral, ongoing part of the teaching-learning process, conducted with a view to improving student learning.

Evaluation and assessment are two interrelated but distinct components of this process:

- *Evaluation* involves subjective judgement against selected criteria, based on relevant data.
- *Assessment* is the collection of the data on which the evaluation depends.

The effective evaluation of student achievement will:

- provide the student with information regarding what has been learned and what requires further study;
- provide the teacher with information on which to base the continuation of the teaching-learning cycle;
- provide parents with information regarding their child's progress;
- provide teachers at the next level or others with information concerning the course objectives achieved;
- encourage the student to further achievement.

The effective evaluation of the program will verify the appropriateness of stated goals and objectives and determine the effectiveness of the teaching strategies, materials, and assessment instruments that have been used.

Measuring, assessing, or testing provides only the basis for professional judgements made by:

- the teacher on what to report to students, parents, and others and on how to continue the teaching-learning cycle;
- teachers, principals, and supervisory personnel on how to improve program effectiveness.

Evaluation of Student Achievement

Evaluation of student achievement is the process of making judgements about student progress in relation to curriculum goals, aims, and objectives, using information gathered by assessment activities.

Formative evaluation. Formative evaluation of student achievement takes place while students are studying and learning new material. It should be used to inform students of their progress and to identify for the teacher those objectives that require review as well as those that require further and perhaps different instructional attention. Written diagnostic tests or oral questioning of students on prerequisite or current content objectives are important components of formative evaluation. If a pretest indicates that a student or group of students already has mastered much of the material intended to be taught, the teacher should introduce only the unknown skills, concepts, or generalizations and emphasize the application of these skills, concepts, and generalizations to problem solving. The following are sources of data on which formative evaluation may be based:

- records of previous achievement
- topic or unit pretests (ranging from informal oral questioning to written diagnostic tests)
- systematic observations of attitudes and performance, including oral and written presentations of solutions to problems
- informal questioning related to specific objectives
- self-evaluation by students
- teacher-student discussions or interviews, which could include informal or formal diagnostic strategies to probe levels of understanding

Summative evaluation. Summative evaluation of student achievement takes place after the completion of a topic or unit. It should be used to inform students and others of the extent to which the instructional objectives have been achieved. The following are sources of data on which summative evaluation may be based:

- topic or unit post-tests
- formal examinations at specific intervals
- standardized tests *related to course objectives*
- projects

Although the teaching of mathematics is usually sequential, the learning of mathematics is often not sequential and is not easy to measure. Assessment practices shall reflect the complexity of learning. They shall deal with degrees of understanding, with processes such as problem solving and proof, and with values, attitudes, and effort as well as facts, skills, concepts, and generalizations. Written tests or examinations, no matter how well constructed, do not provide all of the information needed for a valid evaluation. It is only after the information provided by discussion among students, interviews, teacher observations, projects, tests, and examinations has been analysed and studied that a sound professional judgement can be made. Whatever form of assessment is used, care should be taken to ensure that it *reflects the instruction given*. This will be better accomplished through criterion-referenced testing (assessment on specific objectives) than through norm-referenced testing (assessment based on a comparison with the achievement of peers).

Most of the teacher's assessments, particularly for students enrolled in basic- and general-level mathematics courses, should be formative, aimed at providing individuals or groups of students with information concerning their progress over the short term. The teacher can then decide whether or not the learning situation has been effective and can provide for changes in pacing, materials, teaching methods, or objectives.

Reports on the progress of students should be made at regular intervals. To contribute fully to the improvement of learning, any report of progress should be interpreted so that the student understands the meaning of the evaluation. Where possible, reports to parents and others should also be interpreted through an interview or letter so that the implications of the evaluation are understood.

The design of assessment instruments should reflect the relative emphasis within a topic, unit, or course on cognitive categories, such as recognition and recall, use of algorithms, comprehension, application, and problem solving.

A well-designed post-test or examination should provide motivation for and assessment of the synthesis of a variety of knowledge and process objectives. It should not be limited to the assessment of isolated objectives.

End-of-course evaluation should represent a judgement based on formative and summative assessments. The evaluation, particularly at the interfaces of Grades 8 and 9, Grade 12 and colleges of applied arts and technology, and the OACs and university, should reflect the achievement of the student on the core objectives of the course and should be an indication of the student's readiness for further study in mathematics. Consistency of evaluation among teachers, schools, and boards is of particular importance at these interfaces.

Evaluation depends on decisions concerning standards of achievement. Standards cannot be established as absolute and applicable to all students without denying the wide range of differences and circumstances that affect learning. Standards of achievement shall depend on age, ability, and aptitude. In establishing standards of achievement for students, particularly those in the Senior Division, teachers shall consider the expectations of post-secondary institutions and the community. The standards should be high enough to enhance the quality of education, but not so high as to discourage further learning. Where standards require a demonstration of mastery of material based on criteria such as four correct answers in five attempts, the assessment procedures should include a test-retest cycle, with appropriate opportunities for students to relearn the material.

Exceptional students may require specially designed evaluation strategies. (See the subsection on exceptional students in the section entitled "Cross-Curricular Components" above.)

There is a specific statement regarding the evaluation of student achievement in the introduction to each pair of courses outlined in this guideline.

Evaluation of Program Effectiveness

The evaluation of program effectiveness is the process of making judgements about the success of the program in achieving the curriculum goals, using information gathered through a variety of assessment procedures. Both formative and summative evaluation of student achievement provide assessment data for the evaluation of program effectiveness, but other data should be collected by teachers, principals, and supervisory personnel as a basis for such evaluation. The following are sources of such data:

- records of teaching strategies used for different topics and of the amount of time spent on different topics
- classroom observation of teaching strategies and of the time spent on specific strategies
- questionnaires that sample teachers' goals, strategies, and emphases
- comparisons of local curriculum objectives and guideline objectives and of the objectives of resources (print and other) and local curriculum objectives
- questionnaires and interviews that provide samples of community goals and standards
- criteria used for recommending program choices to students (These criteria should be modified to accommodate the needs of individuals.)
- records of the provision of, and teachers' participation in, professional activities related to curriculum objectives
- the analysis of student-assessment instruments and student achievement on them as related to local curriculum objectives
- the analysis of the availability and use of resources such as manipulative materials, calculators, and computers
- samples of student achievement on selected objectives
- records of student achievement in other schools, boards, provinces, or countries
- enrolment data for sequences of courses over a number of years

The following are some questions that may be asked as part of the program-evaluation process:

- How well does the local curriculum match the ministry guideline in both process and content objectives?
- How well does student achievement match community expectations?
- Are the courses meeting the needs, interests, and abilities of students?
- Is the sequencing of topics appropriate?
- Should the amount of time devoted to, or the depth of treatment of, a topic be changed?
- Is there a need for in-service programs or resource documents?
- Are the objectives of the course appropriate?

The result of program evaluation should be recommendations concerning the following:

- teaching strategies
- evaluation and assessment practices
- curriculum guidelines
- resources
- professional development
- appropriateness and maintenance of standards
- appropriateness of, and suggestions for changes in, course objectives and structure
- provision for the individual needs, interests, or abilities of students

The evaluation of program effectiveness should be systematically undertaken at the classroom, school, board, and provincial levels. As with the evaluation of student achievement, the evaluation of program effectiveness should contribute to the creation of a learning environment in which each student is encouraged to strive for full realization of his/her potential.

ONTERIS

Supervisory personnel, principals, department heads, and teachers will find it useful to identify sources of support materials through ONTERIS (Ontario Educational Resources Information System). The data base includes records of research reports, curriculum guidelines, school board courses of study, curriculum resource documents, and teachers' reference manuals. These materials will provide additional assistance to teachers and school board teams preparing local courses of study.

Educators are encouraged to submit documents originating at the local or board level for recording in the data base. Such documents should be submitted through the channel established by the board.

ONTERIS may be accessed via a terminal or microcomputer through the BRS Information Technologies, a database search vendor. Further information about ONTERIS may be obtained from:

ONTERIS
Computers in Education Centre
Ministry of Education
Mowat Block, 13th Floor
Queen's Park
Toronto, Ontario
M7A 1L2

Mathematics, Grades 7 and 8

Introduction

Aims

Courses in Mathematics, Grade 7 and Mathematics, Grade 8 should be designed to assist each student to:

- develop a positive attitude towards mathematics;
- develop an appreciation for the place of mathematics, and its widespread applications, in our culture;
- develop problem-solving abilities;
- develop facility in communication skills involving the use of the language and notation of mathematics;
- develop facility in the appropriate use of the calculator;
- consolidate and extend his/her skills in arithmetic, algebra, measurement, and geometry;
- participate in those experiences that will lay a foundation for the development of the more abstract features of mathematics in later grades.

Process Components

Experiential approach. Students in Grades 7 and 8 are at different levels in their understanding of mathematics and in the development of associated skills. They also differ in their attitude towards the subject. An experiential approach will help to arouse and maintain their curiosity, should allow each student to work successfully at his/her own level, and should help keep the students' anxiety level to a minimum. Students should learn to record systematically the results of their investigations and to use these recordings to search for patterns and relationships. Because most students in the first two years of the Intermediate Division are not yet able to think abstractly, optimum use should be made of concrete manipulative aids in activities and investigations. The experiential approach is particularly important to the understanding of concepts related to fractions, ratio, percent, decimals, algebra, measurement, and geometry.

Proof. Most students in Grades 7 and 8 have difficulty understanding the process of, or even the need for, formal proof. For some, challenges to justify statements, search for other cases, or look for counter examples are experiences that will help them develop an appreciation of proof. All students should be encouraged to develop solutions that are clear and sequential. Attention should be drawn to the differences between problem-solving strategies and the communication of a solution to others.

Problem solving. Emphasis should be placed on the following problem-solving procedures:

1. Identifying relevant and irrelevant information. This involves reading, understanding, and paraphrasing.
2. a) Identifying possible strategies. These include:
    - classifying information (e.g., as insufficient, conflicting, extraneous, redundant);
    - searching for a pattern;
    - drawing a diagram or flow chart;
    - constructing a table;
    - estimating (guessing and checking, improving the guess);
    - choosing mathematical operations and sequencing them;
    - assuming a solution and working backwards;
    - solving a simpler problem;
    - making an assumption and drawing a conclusion.

   b) Selecting a strategy based on familiarity or ease of implementation.
3. Carrying out the strategy as follows:
    - working with care;
    - checking work;
    - presenting ideas clearly;
    - persisting.
4. Determining how good the solution is by verifying it in the problem situation (reasonableness of results).

Evaluation of Student Achievement

Because Grade 7 and 8 students are at different levels in their understanding of mathematics, particular attention must be given to formative evaluation. Assessment techniques such as pretests, diagnostic tests and interviews, and systematic observations should be emphasized. Summative evaluation should concentrate on the core objectives of the courses with particular attention directed to reflecting the relative emphases within the course. Systematic observations and topic and unit post-tests should provide most of the assessment data. Formal examinations in mathematics have little or no place in the assessment of Grade 7 and 8 students. (See the section "Evaluation of Student Achievement", page 22.)

Relative Emphasis

The following charts present a suggested time frame for the different sections of the courses and indicate the relative emphasis to be placed on each. Since many of the course objectives can best be achieved through an integration of the material, this time frame should not be considered prescriptive. Only the core objectives and the process components related to them have been considered in establishing this time frame.

Mathematics, Grade 7

Section	*Title*		*Suggested Number of Hours*
	Numerical Methods and Algebra		
1.	Calculator Skills		2
2.	Whole Numbers and Decimals		18
3.	Factors		2
4.	Fractions		10
5.	Percent		12
6.	Integers		5
7.	Organization and Representation of Data		7
8.	Mathematics and the Computer		optional
9.	From Arithmetic to Algebra		8
	Measurement and Geometry		
1.	Measurement		10
2.	Basic Notions in Geometry		10
3.	Three-Dimensional Geometry		6
4.	Transformations		10
	Core extension or options	Minimum	20
		Total Minimum	120

Mathematics, Grade 8

Section	*Title*		*Suggested Number of Hours*
	Numerical Methods and Algebra		
1.	Calculator Skills		2
2.	Whole Numbers and Decimals		8
3.	Factors, Powers, Square Root		6
4.	Fractions and Decimals		7
5.	Ratio and Rate		8
6.	Percent		8
7.	Integers		7
8.	Organization and Representation of Data		7
9.	Variables, Formulas, and Equations		9
10.	Relations		optional
11.	Mathematics and the Computer		optional
12.	Probability		optional
	Measurement and Geometry		
1.	Measurement		11
2.	Constructions		10
3.	Properties of Plane Figures		6
4.	Three-Dimensional Geometry		5
5.	Transformations		6
	Core extension or options	Minimum	20
		Total Minimum	120

Note: Courses should be designed to integrate appropriate objectives from different sections. The order of the sections is not to be interpreted as recommending a teaching sequence. "Measurement and Geometry" objectives should be given attention throughout the year and not left until all other objectives have been dealt with.

Mathematics, Grade 7

In the development of a Grade 7 mathematics course, careful consideration shall be given to the role of experiential methods, modelling, problem solving, and applications. The course shall be designed so that these processes are integrated throughout.

Note: In the following list of content objectives, items are identified as review (R), review and extension (E), or new work (N), on the basis of an interpretation of *The Formative Years*.

Numerical Methods and Algebra

1. Calculator Skills

Good procedures should be developed so that students can use calculators with confidence in problem-solving activities. The use of the constant feature should be taught when it is available on the calculators being used. These skills should be taught in combination with appropriate topics from other sections.

N/E	a)	Clearing the display of a calculator; correcting entry errors
N/E	b)	Using calculators to add, subtract, multiply, and divide
optional	c)	Using the constant feature in computations
N/E	d)	Selecting from calculator displays the number of decimal places appropriate to the context of a calculation
N/E	e)	Using rounding and estimation
N/E	f)	Checking the reasonableness of an answer obtained with calculators

2. Whole Numbers and Decimals

The work with whole numbers and decimals should emphasize the development and consolidation of appropriate computational skills in the setting of applications and problem solving. Activities in other subject areas and other topics in this guideline should be integrated with this section. For computations beyond those identified in (d) to (g) below, calculators shall be used. Estimation and mental computation should be used to anticipate and check calculator results.

R	a)	Identifying whole-number place values to hundred thousands
R	b)	Identifying decimal place values to thousandths
E	c)	Expressing a number in expanded form employing 1000, 100, 10, 1, 0.1, 0.01, 0.001
R	d)	Adding whole numbers and decimals without calculators (up to four digits and three addends); subtracting whole numbers and decimals without calculators (up to five digits)
R	e)	Multiplying whole numbers and decimals without calculators (up to three digits by three digits)
R	f)	Dividing whole numbers and decimals without calculators (up to five digits by a divisor of two digits); short division (up to four digits by a divisor with one non-zero digit)

E g) Multiplying and dividing a number by 10, 100, 1000
N h) Expressing powers of 10 in exponential form
E i) Applying operations with whole numbers and decimals using calculators
E j) Applying estimation, rounding, reasonableness of results in calculations, in problem solving, and in applications
E k) Calculating average (mean); applications

3. Factors

This section is intended as a brief introduction to the basic terminology and techniques of factoring. A basis should be established for the work with equivalent fractions in section 4(b) below and for further work with factors and fractions in Grade 8, "Numerical Methods and Algebra", sections 3 and 4.

E a) Identifying factors of a number
N b) Determining factors of a number up to 50 without calculators
optional c) Determining factors of a number up to 100 with calculators

4. Fractions

Students' understanding of the concepts of fractions and operations with fractions should be reviewed by means of objects, pictures, and diagrams. This link between the fractional symbols and manipulative materials or pictorial representations should be maintained throughout the Intermediate Division. The use of equivalent fractions in addition and subtraction should be stressed. Operations with mixed numbers should not be emphasized and in most applications should be dealt with by changing the mixed numbers to decimals. Applications and problem solving shall be used to reinforce the skills reviewed and developed. Since the focus is to be on fractions rather than on computation with whole numbers, simple denominators such as 2, 3, 4, 5, 8, and 10 should be used.

E a) Interpreting a fraction as part of a unit, as part of a set, as division; expressing one-half, quarters, fifths, and tenths as decimals
E b) Identifying and determining equivalent fractions
E c) Ordering fractions using models or equivalent fractions
E d) Identifying and determining reciprocals of fractions
E e) Adding, subtracting, multiplying, and dividing proper and improper fractions (two fractions per question)
E f) Interpreting mixed numbers in applications; converting mixed numbers to decimals with and without calculators

5. Percent

This section should involve concrete experiences with commonly used whole-number percentages and their relation to decimals and fractions. The emphasis should be on problem solving involving simple percentage problems related to student experiences. Students should make appropriate use of calculators in the solution of problems.

N a) Expressing whole-number percents as decimals and as fractions
N b) Expressing one-half, quarters, tenths, and fifths as percents
N c) Expressing decimals as percents
N d) Calculating a percentage of a number; calculating what percent one number is of another
N e) Applying percents to practical problems (e.g., sales tax, simple interest, discount)

6. Integers

Familiar examples should be used to consolidate the concept of integers, to develop a sense of order, and to illustrate addition. Students should develop sufficient skill in the addition of integers to provide them with the basis for the learning of subtraction in Grade 8.

N a) Establishing the need for integers from practical examples
N b) Locating integers on a number line; ordering integers
N c) Adding one-digit or two-digit integers without calculators in the context of practical examples

7. Organization and Representation of Data

Students shall be given opportunities to collect, record, display, and interpret data. Where feasible, this process should be integrated with projects in other subjects. Some discussion of the relative suitability of different types of displays of data for different purposes could be initiated.

E a) Collecting data; recording data
E b) Representing numerical data by bar graphs and broken-line graphs
E c) Reading and interpreting pictographs, bar graphs, and broken-line graphs

8. Mathematics and the Computer

optional a) Using a computer in arithmetic applications
optional b) Investigating the rules that a specific computer language uses to evaluate numerical expressions
optional c) Using a computer to display data in a variety of forms
optional d) Writing simple programs
optional e) Discussing uses of the computer and its impact on individuals and society

9. From Arithmetic to Algebra

Algebraic notation should be closely related to arithmetic and to formulas (see "Measurement and Geometry", section 1). The emphasis should be on interpreting algebraic expressions and statements as generalizations of numerical ones in situations and problems involving applications familiar to students. This link between algebra and arithmetic should be further strengthened through the solution of equations by *systematic trial*.

N	a)	Using brackets in writing and interpreting numerical expressions; evaluating numerical expressions with and without brackets, using rules for the order of operations
N	b)	Evaluating simple algebraic expressions by substitution
N	c)	Translating word phrases and statements into algebraic expressions or equations
N	d)	Interpreting algebraic expressions and equations as word phrases or statements
N	e)	Solving, by systematic trial, with and without calculators, equations of the form $ax = c$, $ax + b = c$ involving whole numbers

Measurement and Geometry

1. Measurement

Students should estimate the size of, and measure, large and small objects. Measurement should be related to other sections of the course as well as being incorporated, where possible, into projects in other subjects. In the process, formulas should be developed as a logical consequence of activities in a progression from the concrete to the abstract. Attention should be given to consolidating the work with the metric system that has been developed in previous grades.

E	a)	Estimating and measuring lengths of line segments, lengths of portions of curves, and perimeters of figures
E	b)	Calculating in numerical cases perimeters of figures bounded by line segments; generalizing statements for some figures
N	c)	Stating and applying a formula for the perimeter of a rectangle
E	d)	Estimating and measuring the areas of regular and irregular polygons
E	e)	Measuring the area of irregular figures by means of a square grid
E	f)	Calculating in numerical cases the area of a square and of a rectangle; generalizing to formulas
N	g)	Calculating in numerical cases the area of a triangle and of a parallelogram
E	h)	Measuring or calculating length, perimeter, and area in applications
R	i)	Measuring angles with a protractor
E	j)	Estimating and calculating the volume of a rectangular prism; generalizing to a formula
E	k)	Converting measurements among commonly used metric units
optional	l)	Measuring and calculating the volume of a triangular prism

2. Basic Notions in Geometry

The emphasis should be on clarifying the classifications identified in earlier grades, making specific the geometric properties related to the classifications, and using correct terminology. The activities of "Measurement and Geometry", sections 3 and 4, and the constructions in this section should be used to identify and reinforce the classifications and the properties.

Construction instruments should not be limited solely to the traditional ruler and compasses. Students should be exposed to other techniques that reinforce the recognition of properties not evident when ruler and compasses are used exclusively.

E	a)	Classifying angles; identifying acute, right, obtuse, and straight angles
E	b)	Classifying pairs of lines; identifying properties of intersecting, perpendicular, and parallel lines
E	c)	Classifying triangles according to angles, sides, and symmetry
E	d)	Classifying quadrilaterals according to angles, sides, and symmetry; identifying squares, rectangles, parallelograms, rhombuses, and trapezoids
E	e)	Identifying congruent figures; identifying properties of congruent figures (size, shape, angles, lengths)
E	f)	Identifying line symmetry and rotational symmetry of regular polygons; identifying the congruence of corresponding parts
N	g)	Constructing geometric figures (congruent segments, congruent angles, perpendicular lines, parallel lines, perpendicular bisector, angle bisector, equilateral triangle, isosceles triangle, square, rectangle, parallelogram) using geometric properties and a variety of techniques and instruments (ruler, protractor, compasses, paper folding, tracing paper, transparent mirror)
E	h)	Identifying geometric properties in designs; applying geometric properties in drawing designs
E	i)	Graphing ordered pairs in the first quadrant

3. Three-Dimensional Geometry

Students should have the opportunity to develop the skills of this section through the use of actual three-dimensional objects, not pictures of them. This section may be related to topics in visual arts or other subjects.

E a) Classifying and identifying polyhedrons (rectangular solids, cubes, triangular prisms, triangular pyramids, square pyramids)
E b) Identifying congruent edges and faces, and parallel and perpendicular edges and faces, in examples of polyhedrons
E c) Constructing regular polyhedrons from their nets
N d) Recognizing three-dimensional objects from sketches (perspective; front, side, and top views)
N e) Sketching front, side, and top views of three-dimensional objects
optional f) Identifying planes of symmetry of three-dimensional objects
optional g) Classifying real-world objects as solids, shells, or skeletons

4. Transformations

The pattern recognition provided by these experiences gives students an opportunity to develop visual perception and reinforces the identification of properties in "Measurement and Geometry", section 2. The activities involved in this topic may be related to visual arts or other subjects. Computer graphics could be used, if available.

R a) Sketching the images of a figure under slides, flips, and turns by means of templates or tracing
E b) Recognizing the image of a figure under a slide (translation), flip (reflection), and turn (rotation)
E c) Constructing tiling patterns with congruent tiles
E d) Identifying properties of figures in tiling patterns
E e) Identifying translations, reflections, and rotations in tiling patterns
N f) Identifying the basic properties of translation, reflection, and rotation
N g) Drawing translation, rotation, and reflection images using a method that emphasizes the basic property for each transformation
E h) Solving tiling puzzles (tangram and polyomino puzzles); identifying symmetries (line, rotational) in the tiles and figures
N i) Making enlargements and reductions of figures using a geoboard, dot paper, a grid, or tiles
optional j) Identifying relationships between line symmetry and reflection, rotational symmetry and rotation
optional k) Drawing images of figures under distortion using approaches such as distorted grids, non-orthogonal axes, or computer graphics

Mathematics, Grade 8

In the development of a Grade 8 mathematics course, careful consideration shall be given to the role of experiential methods, modelling, problem solving, and applications. The course shall be designed so that these processes are integrated throughout.

Note: In the following list of content objectives, items are identified as review (R), review and extension (E), or new work (N), on the basis of an interpretation of *The Formative Years* and the Mathematics, Grade 7 list of content objectives.

Numerical Methods and Algebra

1. Calculator Skills

Good procedures should be reviewed and extended so that students can use calculators with confidence in problem-solving activities. The use of the constant feature should be taught when it is available on the calculators being used. These skills should be taught in combination with appropriate topics from other sections.

N	a)	Identifying the appropriate and inappropriate uses of calculators
R	b)	Clearing the display of a calculator; correcting entry errors
R	c)	Using calculators to add, subtract, multiply, and divide
optional	d)	Using the constant feature in computations
N	e)	Using subtotals and memory for adding, subtracting, multiplying, and dividing
N	f)	Ordering operations to obtain a desired result with calculators
R	g)	Selecting from calculator displays the number of decimal places appropriate to the context of a calculation
E	h)	Using rounding and estimation
R	i)	Checking the reasonableness of an answer obtained with calculators

2. Whole Numbers and Decimals

The work on place value consolidates and extends that of Grade 7. Computational skills should be reviewed and practised in the setting of applications and problem solving with an appropriate use of estimation and mental computation to anticipate the reasonableness of results. Help should be given to students who have not yet mastered those skills that have been identified as to be done without a calculator (see Grade 7, "Numerical Methods and Algebra", section 2). For computations beyond those identified, calculators shall be used. Problem solving and number patterns should be used to introduce some of the concepts in "Numerical Methods and Algebra", section 9, in conjunction with topics from "Measurement and Geometry", section 1. Appropriate topics from"Numerical Methods and Algebra", sections 3, 4, 6, and 8 should be combined with this section. Topics in science, geography, or other subjects could provide opportunities for applications.

E	a)	Identifying whole-number place values to millions
R	b)	Identifying decimal place values to thousandths
E	c)	Expressing a number in expanded form employing $10^6, \ldots, 10^2, 10, 1, \frac{1}{10}, \frac{1}{10^2}$, and $\frac{1}{10^3}$
E	d)	Consolidating and applying the addition, subtraction, multiplication, and division of whole numbers and decimals with and without calculators
E	e)	Multiplying and dividing a number mentally by 10, 100, and 1000
R	f)	Evaluating numerical expressions using rules for the order of operations
E	g)	Applying estimation, rounding, and reasonableness of results in calculations, in problem solving, and in applications
optional	h)	Identifying rounding errors resulting from successive calculations, in particular with a calculator

3. Factors, Powers, Square Root

This section should extend the concept of factor (Grade 7, "Numerical Methods and Algebra", section 3) so that computation with fractions and ratios ("Numerical Methods and Algebra", sections 4 and 5) can be handled efficiently. The emphasis should be on the recognition of factors rather than on the use of algorithms to generate all factors. The occurrence of repeated factors leads to the use of the exponential form. Some calculator and computer displays could be used to illustrate scientific notation for large numbers. The concept of the square of a number can be related to both exponential notation and geometry. The inverse concept, square root, should be related to the problem of finding the length of the side of a square when given the area. The Pythagorean theorem could be introduced in conjunction with this topic; see "Measurement and Geometry", section 3(e).

E	a)	Identifying and applying divisibility tests for 2, 3, 5, 9, and 10
E	b)	Identifying the factors of a number; identifying prime numbers and composite numbers
E	c)	Determining factors of a composite number up to 100 without calculators
N	d)	Writing repeated factors in exponential form; writing numbers in exponential form as repeated factors
N	e)	Reading and interpreting large numbers in scientific notation as they occur in calculator or computer displays
E	f)	Identifying the squares of whole numbers from 1 to 10; identifying the square roots of perfect squares from 1 to 100
N	g)	Finding an approximation of the square root of whole numbers using calculators

4. Fractions and Decimals

The algorithms for addition, subtraction, multiplication, and division should be consolidated in this section. Simple denominators such as 2, 3, 4, 5, 8, and 10 should be used; questions involving more than four fractions should be avoided. Mixed numbers should be dealt with, in most cases, by converting them to decimals. Investigations to determine the complete period of the decimal form of some fractions can generate challenging problems in the use of calculators. Students should develop a facility in converting terminating decimals to fractions, reducing to lowest terms in the simpler cases.

E	a)	Adding, subtracting, multiplying, and dividing fractions with simple denominators (up to three operations per question); applications
E	b)	Converting fractions and mixed numbers to decimals, with and without calculators
N	c)	Demonstrating that the decimal form of a fraction either terminates or repeats
E	d)	Converting terminating decimals to fractions
optional	e)	Investigating the patterns of repeating decimals with calculators

5. Ratio and Rate

This is the introduction to ratio and rate for most students. Informal language should be used rather than formal notation. The concept of ratio as a comparison of two or more quantities should be introduced in the context of applications familiar to students. Students should generate tables indicating the variety of pairs or triples associated with a given ratio. Problems should be solved through the use of the unitary approach or equivalent ratio ideas before the fractional notation for two-termed ratios is introduced. Concrete materials should be used when students have difficulty. Scale drawings – see "Measurement and Geometry", section 5(c) – should be related to the work of this section. Formal algebraic solutions of fractional equations are not to be used except in situations where a course has been modified as an enriched offering with a substantial increase in algebraic content.

N	a)	Identifying a ratio as the comparison of two or more quantities in the same units
N	b)	Expressing two-termed ratios in fractional form; distinguishing between fractions and two-termed ratios in fractional form
N	c)	Stating equivalent ratios involving two or three terms; simple applications of equivalent ratios
N	d)	Identifying a rate as a comparison of two quantities in different units; simple applications of rates in problem solving (e.g., exercise rates, cost per unit)

6. Percent

The concept of percent as related to decimals and fractions is to be consolidated with the introduction of percents greater than one hundred and less than one. The relation of percent to ratio should be introduced but not emphasized. Problems related to students' experiences and common uses of percent should be emphasized. Students should make appropriate use of calculators.

E	a)	Expressing percents (including percents greater than one hundred or less than one) as decimals, as fractions, or as ratios
E	b)	Expressing decimals, fractions, and ratios as percents
E	c)	Applying percents to practical problems such as sales tax, simple interest, commission, and discount

7. Integers

The concept of integers, the ordering of integers, and the addition of integers should be consolidated in the context of real-life examples. Subtraction should be introduced with appropriate imagery to develop students' understanding of the process rather than their rote application of a rule. The graphing of ordered pairs should be extended to pairs of integers. If multiplication of integers is taught, the emphasis should be on pictorial representation or patterns, that is, on providing a basis for the rules rather than practice with the rules.

R	a)	Identifying the use of integers in practical examples
R	b)	Locating integers on the number line; ordering integers
E	c)	Locating ordered pairs of integers in the plane
E	d)	Adding and subtracting one-digit and two-digit integers without calculators
optional	e)	Multiplying and dividing one-digit and two-digit integers without calculators

8. Organization and Representation of Data

Students should be given opportunities to collect, record, display, and interpret data. Pictographs should be extended beyond the simple cases considered in previous grades. Circle graphs should be associated with percent – see "Numerical Methods and Algebra", section 6 – and with protractor constructions – see "Measurement and Geometry", section 2(a). Projects related to other subjects should be used to consolidate the skills of this section.

E	a)	Collecting data; organizing data
E	b)	Representing data by bar graphs and broken-line graphs
E	c)	Reading and interpreting bar graphs and broken-line graphs
E	d)	Reading and interpreting pictographs in which one element represents more than one object
N	e)	Representing data by circle graphs
N	f)	Reading and interpreting circle graphs
optional	g)	Identifying distortion of the representation of data by techniques such as the use of scales and pictographs

9. Variables, Formulas, and Equations

Number patterns, arithmetic problems, and measurement formulas should be used to review the use of algebraic notation. The emphasis should be placed on the interpretation of algebraic expressions and equations as generalizations of arithmetical expressions and equations. Informal but systematic methods of solving equations should be consolidated.

E	a)	Evaluating algebraic expressions by substitution with and without calculators
optional	b)	Evaluating algebraic expressions by substituting integers
N	c)	Developing simple formulas by generalizing from numerical cases
E	d)	Translating word phrases and statements to algebraic expressions or equations
E	e)	Interpreting algebraic expressions and equations related to applications as word phrases or statements
optional	f)	Representing sequential activities by flow charts; representing algorithms for calculators by flow charts
E	g)	Solving linear equations in one variable by informal methods such as inspection, systematic trial, or inverse flow charting
E	h)	Substituting in formulas and solving the resulting linear equations with and without calculators

10. Relations

optional	a)	Representing non-numerical and numerical relationships by arrow diagrams, mappings, and ordered pairs
optional	b)	Representing simple relations using ordered pairs; graphing these relations

11. Mathematics and the Computer

optional	a)	Using prepared programs related to numerical methods, algebra, measurement, or geometry and their applications; simulations
optional	b)	Writing simple programs
optional	c)	Discussing uses and misuses of computers and their impact on individuals and society

12. Probability

optional	a)	Listing the possible outcomes of simple experiments
optional	b)	Identifying the favourable outcomes among the possible outcomes; determining the associated probability
optional	c)	Comparing predicted and experimental results
optional	d)	Using sets and Venn diagrams to represent and analyse different outcomes of an experiment

Measurement and Geometry

1. Measurement

Estimation and measurement associated with real objects should be practised. The formulas introduced in Grade 7 ("Measurement and Geometry", section 1) should be used in applications. The formulas for circumference and area of a circle and the volume of non-rectangular prisms should be introduced through activities. This topic could be related to associated topics in science or other subjects.

R	a)	Consolidating conversions among commonly used units within the metric system
R	b)	Calculating in numerical cases the perimeter of figures bounded by line segments
E	c)	Calculating in numerical cases the area of irregular figures that can be decomposed into rectangles
E	d)	Calculating the perimeter and area of rectangles, triangles, and parallelograms using appropriate formulas; applications
N	e)	Investigating the relationship between the circumference and the diameter for circular objects to determine an approximation of π
N	f)	Investigating the relationship between the area and the square of the radius for circular objects
N	g)	Stating and applying formulas for the circumference and area of a circle
N	h)	Calculating the surface area of rectangular and triangular prisms by relating it to faces or nets
E	i)	Stating and applying a formula for the volume of a rectangular prism
N	j)	Investigating and calculating the volume of non-rectangular prisms; generalizing the volume of a prism as the product of the height and the area of its base; applications
optional	k)	Determining the surface area and volume of a cylinder

2. Constructions

Students should use a variety of construction instruments and techniques. The emphasis should be placed on relating the properties of figures to constructions. Generalizations should be clearly identified and related to "Measurement and Geometry", section 3.

R	a)	Constructing congruent segments and angles, perpendicular lines, parallel lines, perpendicular bisectors, and angle bisectors, using geometric properties and a variety of techniques and materials (ruler, protractor, compasses, paper folding, tracing paper, transparent mirror)
E	b)	Testing for perpendicular lines (paper folding, set square, protractor, grid paper, transparent mirror)
E	c)	Testing for parallel lines (parallel ruler, grid paper, ruler and set square, paper folding, ruler, transparent mirror, alternate or corresponding angles, protractor)
N	d)	Constructing 90°, 60°, 45°, and 30° angles without a protractor
E	e)	Constructing triangles and quadrilaterals from given information
N	f)	Constructing a triangle congruent to a given triangle; investigating the conditions for congruent triangles
N	g)	Constructing a circle given its centre and radius, centre and diameter, or centre and a point of the circle
N	h)	Locating the centre of a circular arc by paper folding or transparent mirror

3. Properties of Plane Figures

The properties of plane figures should be identified through measurement and construction activities (see "Measurement and Geometry", sections 1 and 2). Clear statements should be used to record the properties identified in order to assist students in recalling them in later grades. A consolidation of the knowledge of these properties is provided through their application to numerical problems.

E	a)	Identifying through investigation the equal- and supplementary-angle properties of intersecting lines and parallel lines
E	b)	Identifying through investigation the equal-angle properties of isosceles and equilateral triangles
E	c)	Identifying through investigation the sum-of-the-angles property of a triangle
E	d)	Applying the above properties in numerical cases
N	e)	Investigating the Pythagorean theorem; numerical applications
optional	f)	Investigating the rigidity or non-rigidity of polygonal figures
optional	g)	Identifying the symmetries (line symmetry, rotational symmetry) and angle properties of regular polygons

4. Three-Dimensional Geometry

The emphasis should be placed on the development of students' visual perception through experience. The sketching involved should not stress formal drafting conventions, but should reinforce the development of students' ability to recognize different views of objects.

E a) Constructing skeletons of pyramids and prisms; classifying pyramids and prisms by their bases
E b) Recognizing three-dimensional objects from sketches (perspective; front, side, and top views)
N c) Sketching simple three-dimensional objects (e.g., rectangular solids, pyramids, cylinders, cones)
optional d) Identifying the symmetries (line, plane, rotational, point) of three-dimensional objects
optional e) Constructing regular polyhedrons from their nets; stacking solids in space
optional f) Investigating the rigidity or non-rigidity of polyhedral skeletons

5. Transformations

The use of the basic properties of translation, reflection, and rotation should be consolidated. Dilatations should be introduced in the context of scale drawings. The properties of similar figures should be identified and generalized in clear statements. The concept of ratio (see "Numerical Methods and Algebra", section 5) should be applied in drawing dilatations and similar figures.

E a) Identifying the image of a figure under translation, reflection, and rotation using the basic property of each transformation; drawing such images
E b) Identifying (using tracing paper, transparent mirrors) the transformation relating two congruent figures; drawing the translation arrow, reflection line, or rotation centre and angle
E c) Drawing enlargements and reductions; making and using scale drawings; concept of scale factor (scale ratio); applications such as maps and floor plans
E d) Identifying the properties of similar figures (shape, sides, angles, area)
optional e) Determining the areas of similar figures in tiling patterns
optional f) Observing and predicting reflections in parallel mirrors and intersecting mirrors; applications to art and special effects in films

Mathematics, Grades 9 and 10, Advanced Level

Introduction

Aims

Courses in Mathematics, Grades 9 and 10, Advanced Level, should be designed to assist each student to:

- develop a positive attitude towards mathematics;
- develop an appreciation for the place of mathematics and its widespread applications in our culture;
- develop problem-solving abilities;
- develop facility in communication skills involving the use of the language and notation of mathematics;
- develop facility in the appropriate use of a calculator;
- consolidate and extend the skills in arithmetic, algebra, measurement, and geometry that have been introduced in previous grades;
- extend his/her ability to handle the more abstract features of mathematics;
- prepare for the Ontario Academic Courses in mathematics.

The advanced-level courses in mathematics for Grades 9 and 10 are designed for students who have a good understanding of Grades 7 and 8 mathematics and have demonstrated interest in and aptitude for the more abstract aspects of these courses.

The emphasis throughout the courses should be placed on developing a foundation for the abstract mathematics of the Senior Division.

Process Components

Experiential approach. The role of experience in providing a firm foundation for abstractions should be recognized. Wherever possible, concepts should be related to experience so that the mental images based on such experience will give meaning to the concepts.

The problems selected should place concepts in contexts that are relevant to students. Every opportunity should be taken to provide students with experiences that integrate the various strands of the course, so that mathematics is perceived as a unity and not as separate subjects (arithmetic, algebra, and geometry).

Proof. Students should develop an appreciation for the processes of proof by being challenged to justify statements, to search for other cases, to look for counter examples, and to demonstrate the existence of a number or a geometric figure with certain properties. Proofs based on listing all possible cases, identifying counter examples, or demonstrating a contradiction should be used. The teacher should model logical thinking through the consistent use of conditional reasoning (e.g., "if . . . then" sentences).

In a Grade 10 advanced-level mathematics course, attention should be directed to the role of the building blocks of a proof (undefined terms, definitions, assumptions), with geometry used as a model for a logical system. Direct proof, as a chain of "if . . . then" statements, although used primarily in the geometry strand, should also be used where appropriate in the algebra and numerical methods strands.

Students' attention should be drawn to the differences between problem-solving strategies and the ultimate structured proof.

In the evaluation of a student's knowledge of proof during the Intermediate Division, it should be remembered that the ability to think hypothetically, although encouraged by experience, appears to be a product of maturation. Evaluation should be designed to encourage growth rather than to discourage experimentation with thought processes that may be foreign to some students.

Problem solving. The emphasis should be placed on the following problem-solving procedures:

1. Identifying relevant and irrelevant information. This involves reading, understanding, paraphrasing, summarizing, and listing.
2. a) Identifying possible strategies. These include:
    - classifying information (e.g., as insufficient, conflicting, extraneous, redundant);
    - searching for a pattern;
    - drawing a diagram or flow chart;
    - constructing a table;
    - estimating (guessing and checking, improving the guess);
    - choosing mathematical operations and sequencing them;
    - assuming a solution and working backwards;
    - using a formula or writing an equation;
    - solving a simpler problem;
    - making an assumption and drawing a conclusion.

   b) Selecting a strategy based on familiarity or ease of implementation.
3. Carrying out the strategy as follows:
    - working with care;
    - checking work;
    - presenting ideas clearly;
    - persisting.
4. Determining how good the solution is by:
    - verifying it in the problem situation (reasonableness of results);
    - searching for a better solution.

Evaluation of Student Achievement

Pretests and systematic observation should be used to identify topics that may be de-emphasized because of prior knowledge of them on the part of students and those that require greater emphasis because of evident lack of knowledge. While data collected through post-tests, systematic observations, and projects should be considered, results on formal examinations shall constitute a significant component of any summative evaluation. Formal examinations should reflect the relative emphases within each course on the various objectives and should include some assessment of problem solving, as well as of recognition and recall, the use of algorithms, and simple applications. (See the section "Evaluation of Student Achievement", page 22.)

Relative Emphasis

The following charts present a suggested time frame for, and indicate the relative emphasis to be placed on, the different sections of the courses. Since many of the course objectives can best be achieved through an integration of the material, this time frame should not be considered prescriptive. Only the core objectives and the process components related to them have been considered in establishing this time frame. Additional optional topics of interest to students may be introduced, provided that the core objectives receive the appropriate emphasis.

Mathematics, Grade 9, Advanced Level

Section	*Title*	*Suggested Number of Hours*
	Numerical Methods	
1.	Applications of Whole Numbers, Decimals, and Integers	5
2.	Powers and Square Roots	7
3.	Fractions and Rational Numbers	6
4.	Applications of Ratio, Rate, and Percent	6
5.	Statistics	8
	Algebra	
1.	Variables and Polynomials	8
2.	Formulas and Equations	10
3.	Relations	5
4.	Probability	optional
5.	Mathematics and the Computer	optional
	Measurement and Geometry	
1.	Measurement and Its Applications	4
2.	Constructions	5
3.	Properties of Plane Figures	6
4.	Coordinate Geometry and Congruence Transformations	6
5.	Dilatation	4
6.	Geometry and Navigation	optional
	Core extension or options	30
	Total	110

Mathematics, Grade 10, Advanced Level

Section	*Title*	*Suggested Number of Hours*
	Numerical Methods	
1.	Powers, Roots, Radicals, and Real Numbers	5
2.	Statistics	8
3.	Relations	4
4.	Matrices	optional
5.	Probability	optional
	Algebra	
1.	Equations and Inequalities	9
2.	Polynomials and Rational Expressions	10
	Geometry	
1.	Basic Notions of Coordinate Geometry	10
2.	The Straight Line	11
3.	Coordinates and Transformations	4
4.	Isometries	optional
5.	Deductive Thinking Through Geometry	13
6.	Three-Dimensional Geometry	6
7.	Vectors	optional
	Core extension or options	30
	Total	110

Mathematics, Grade 9, Advanced Level

Course Code: MAT1A.

Prerequisite: Mathematics, Grade 8.

In the development of a course in Mathematics, Grade 9, Advanced Level, the processes of modelling, problem solving, and applications shall be used in the development of the topics. Concepts that are new to students should be introduced through experiential methods. The general emphasis should be placed on developing mental images that will provide students with a good foundation for generalizations and abstractions appropriate to their mathematical maturity.

Note: In the following lists of content objectives, items are identified as review (R), review and extension (E), or new work (N), on the basis of the listing of content objectives in mathematics for previous grades.

Numerical Methods

1. Applications of Whole Numbers, Decimals, and Integers

The emphasis should be placed on consolidating skills in the context of applications and problem solving. While specific problem-solving techniques may be introduced in some sets of problems, it is essential that an emphasis be placed on the variety of strategies possible for solving most problems.

While calculators shall be used, those computational skills identified in the Grades 7 and 8 mathematics content objectives as to be done without a calculator should be maintained. Students should be given opportunities to practise estimation and mental computation so that they can check calculator results and the reasonableness of solutions to problems. The convention of order of operations should be reviewed as necessary.

The integer topics should be used to consolidate students' skills of addition and subtraction in applied settings and to introduce the multiplication and division of integers. Appropriate topics from section 4 of "Numerical Methods", sections 1 and 2 of "Algebra", and section 1 of "Measurement and Geometry" should be co-ordinated with this section through the use of variables, translating between word phrases and algebraic expressions, generalizing to a formula, substituting in a formula, and measurement problems.

E a) Applying addition, subtraction, multiplication, and division of whole numbers and decimals with and without the use of calculators
E b) Applying estimation, rounding, and reasonableness of results in calculations and in problem solving
E c) Adding and subtracting integers; applications
N d) Multiplying and dividing integers

2. Powers and Square Roots

The use of scientific notation in relation to applications and to some calculator or computer displays should be emphasized. The appropriate and efficient use of a calculator should be emphasized throughout, including the use of the square-root key and exponential-function key when they are available.

E	a)	Writing repeated factors (positive or negative) as powers
E	b)	Evaluating powers with integral bases and whole-number exponents
N	c)	Interpreting and evaluating powers with integral bases and integral exponents
E	d)	Multiplying and dividing numbers by 10, 10^2, 10^3, 10^{-1}, 10^{-2}, and 10^{-3} by inspection
N	e)	Simplifying expressions involving multiplication, division, and powers of powers by writing them as repeated factors
N	f)	Developing exponent laws from numerical cases; simplifying expressions involving multiplication, division, and powers of powers using exponent laws
N	g)	Writing large and small numbers in scientific notation; applications
E	h)	Identifying the squares of integers from -12 to 12
E	i)	Identifying the positive and negative square roots of perfect squares from 1 to 144
N	j)	Estimating to one significant digit the square roots of whole numbers between 1 and 10 000
E	k)	Finding an approximation of the square roots of whole numbers and decimals using a calculator and appropriate rounding
E	l)	Calculating square roots in applied situations
optional	m)	Employing Newton's method for calculating square roots and cube roots using a calculator or computer

3. Fractions and Rational Numbers

The concepts and skills associated with fractions should be extended to rational numbers.

E	a)	Applying the addition, subtraction, multiplication, and division of proper and improper fractions
N	b)	Recognizing that numerals such as $\frac{-1}{2}$, $\frac{1}{-2}$, $-\frac{1}{2}$, and -0.5 represent the same rational number
E	c)	Ordering rational numbers
E	d)	Adding, subtracting, multiplying, and dividing rational numbers (up to three rational numbers per question)
N	e)	Identifying reciprocals as multiplicative inverses
E	f)	Evaluating numerical expressions involving up to four rational numbers using the rules for order of operations
E	g)	Converting rational numbers from fractional form to decimal form with and without the use of calculators
E	h)	Converting terminating decimals (positive and negative) to fractional form
optional	i)	Identifying rounding errors resulting from the use of a calculator or computer for operations with rational numbers; deciding on the acceptability of such errors in applications
optional	j)	Investigating the patterns of repeating decimals with a calculator or computer; stating and testing generalizations based on the investigations

4. Applications of Ratio, Rate, and Percent

Most of this section is review. Where weaknesses are evident in students' retention of the concepts and techniques, reteaching should involve relating the concepts to applications. Sufficient ratios of more than two terms should be involved so that ratio is not solely related to fractional form.

R	a)	Using ratio and rate in the comparison of two or more quantities; expressing rates and two-termed ratios in fractional form
R	b)	Applying equivalent ratios and rates
E	c)	Solving problems involving percents (including percents greater than one hundred or less than one)
R	d)	Interpreting scale drawings and maps; making scale drawings – see "Measurement and Geometry", section 5(c)
optional	e)	Investigating the relationship of percent change in the dimension(s) or perimeter of a rectangle to the percent change in area of the rectangle
optional	f)	Investigating compounding situations (e.g., growth, decay, interest as applications of rate, ratio, and percent)

5. Statistics

Students should be involved in data collection. Illustrations of the use of statistics in the media should be used to provide a basis for a comparison of the methods of data collection with the appropriateness of data and graphs. Data collected by students should be used for the introduction of frequency tables. This form of representation may be unfamiliar to students.

N	a)	Identifying and comparing primary data-gathering methods (e.g., telephone interviews, questionnaires, personal interviews, measurement)
N	b)	Discussing the advantages and disadvantages of data gathered directly and data accepted from other sources

N c) Identifying reasons for sampling and the importance of a good sample

N d) Investigating methods of determining a random sample (draws, random numbers)

optional e) Discussing the advantages and disadvantages of clustered sampling and of stratified sampling

N f) Identifying the suitability of particular sets of data for a specific purpose (appropriateness of data)

E g) Interpreting data presented in tally charts, frequency tables, pictographs, bar graphs, broken-line graphs, and circle graphs

E h) Representing data by means of tally charts, frequency tables, bar graphs, circle graphs, and broken-line graphs

N i) Comparing the suitability of different types of graphs for displaying specific data

N j) Determining the mode, median, and mean

N k) Comparing the appropriateness of the mode, median, and mean as measures of central tendency

optional l) Using techniques such as subtraction of a constant, division by a constant, and assumed mean to simplify the calculation of the mean of a set of values

Algebra

1. Variables and Polynomials

The emphasis should be placed primarily on developing algebra as a modelling strategy. Many of the topics may be linked with topics in "Numerical Methods" and "Measurement and Geometry". The imagery of a variable as representing an arbitrary element of a set of numbers should be stressed and should be associated with formulas and graphs. The role of algebra in communication should be made clear through student experiences in interpreting a formula, in expressing situations in algebraic form, and in translating word phrases and statements into algebra. Having students describe in words situations that might be modelled by expressions such as $3x + 1$ may help them appreciate the role of algebra in modelling. The identified operations with polynomials should be developed carefully and consolidated to provide students with a firm foundation for further algebraic work.

E a) Interpreting a variable as a placeholder

R b) Translating word phrases and statements into algebraic expressions or equations

R c) Interpreting algebraic expressions and equations as word phrases or statements

N d) Graphing simple inequalities in one variable

E e) Evaluating algebraic expressions by substitution

N f) Adding and subtracting polynomials; identifying the concept of additive inverse

N g) Multiplying a monomial, binomial, and trinomial by a monomial

N h) Factoring binomials and trinomials by determining a common factor

N i) Dividing a monomial, binomial, and trinomial by a monomial

2. Formulas and Equations

The association of equations and formulas with applications should be used to emphasize the role of algebra in modelling. The algebraic method should be developed as an extension of the more informal methods of solving equations. Appropriate formulas from other disciplines should be used, particularly in (e) "Manipulating a formula to define a variable explicitly".

E a) Developing formulas from diagrams, numerical charts, and measurement activities

R b) Solving linear equations in one variable by methods such as inspection, systematic trial, and inverse flow charts

N c) Solving linear equations in one variable by the algebraic method

E d) Substituting in a formula involving up to four variables and solving the resulting linear equation

N e) Manipulating a formula to define a variable explicitly rather than implicitly; solving literal equations

N f) Solving problems associated with applications that result in linear equations

N g) Solving linear inequalities in one variable by the algebraic method; graphing the solution of linear inequalities in one variable

3. Relations

Data collected by students for statistics and other disciplines should be used to demonstrate a variety of relations and to provide students with experience in reading graphs.

N a) Representing relations by tables, ordered pairs, or arrow diagrams

N b) Drawing graphs of numerical relations (including non-linear relations) using tables of ordered pairs derived from experiments and realistic situations

N c) Interpolating and extrapolating from graphs, including the reading of decimal coordinates

N d) Investigating examples of linear relations through graphs and tables of ordered pairs; generating related equations

N e) Drawing the graphs of linear relations

4. Probability

optional a) Relating the concept of probability to applications involving the counting of events

optional b) Calculating probability by means of counting and tree diagrams

5. Mathematics and the Computer

optional	a)	Evaluating expressions; producing tables of values, graphs, and so on with the aid of prepared software packages
optional	b)	Using a computer to store and display numerical data in lists, tables, and matrices; locating elements of a table or matrix by row and column; updating lists and tables
optional	c)	Solving problems by means of computer simulations
optional	d)	Writing computer programs that evaluate expressions and produce tables
optional	e)	Discussing some of the uses and misuses of a computer and its impact on individuals and society

Measurement and Geometry

1. Measurement and Its Applications

This section should be associated with appropriate sections of "Numerical Methods" and "Algebra". The emphasis should be placed on the application of concepts and formulas in problem solving. Problems should involve situations in which formulas do not apply (e.g., finding the surface area of an automobile). The optional topics should be taught by means of models and materials.

R	a)	Applying the concepts of perimeter, circumference, and area
R	b)	Applying the formulas for the area of a rectangle, triangle, parallelogram, and circle as well as for the circumference of a circle
optional	c)	Determining the surface area of prisms and cylinders; applications
optional	d)	Determining the volume of prisms and cylinders; applications
optional	e)	Investigating the volume relationship between a pyramid and a prism and between a cone and a cylinder; applications

2. Constructions

The emphasis should be placed on using a variety of techniques and instruments for making constructions (e.g., ruler, protractor, compasses, paper folding, tracing paper, transparent mirrors, set square, grid paper).

R	a)	Constructing congruent angles, angle bisector, perpendicular bisector, and perpendicular to a line (at a point on the line, from a point outside the line)
R	b)	Constructing a circle given its centre and radius, centre and diameter, centre and a point of the circle, or three points of the circle
R	c)	Locating the centre of a circle

3. Properties of Plane Figures

The properties identified through measurement and construction activities in previous grades should be reviewed. An emphasis should be placed on making a precise statement of the properties, since they will provide the basis for deductive proof in Mathematics, Grade 10, Advanced Level. Applications of the properties should be made in numerical problems in order to consolidate them.

R	a)	Stating the equal- and supplementary-angle properties of intersecting lines and parallel lines
R	b)	Stating the equal-angle properties of isosceles and equilateral triangles
R	c)	Stating the sum-of-the-angles property of a triangle
E	d)	Stating the conditions for congruent triangles (sss, sas, asa)
R	e)	Stating the Pythagorean theorem
N	f)	Stating the equidistant properties of points on the perpendicular bisector of a segment and on an angle bisector
N	g)	Identifying through investigation the property relating an inscribed angle of a circle to the angle at the centre subtended by the same arc; identifying the equality property of inscribed angles
N	h)	Identifying through investigation the right-angle property of angles inscribed in a semicircle
N	i)	Identifying through investigation the angle properties of a cyclic quadrilateral
E	j)	Applying the above properties in numerical cases
optional	k)	Investigating the concurrency of lines in a triangle (perpendicular bisectors of sides, angle bisectors, medians) and their relationship to circumcircle, incircle, and centre of gravity

4. Coordinate Geometry and Congruence Transformations

The investigation of transformations through the use of grids provides a basis for the consolidation of the properties of the transformations and the application of coordinate geometry. The basic properties of each transformation should be clearly stated to provide a basis for deductive proof in Mathematics, Grade 10, Advanced Level.

E	a)	Naming the coordinates of points located on a grid; drawing figures on a grid given the coordinates
E	b)	Identifying the image of a figure on a grid under translation, reflection, and rotation; identifying the basic property of each transformation

E	c)	Drawing the image of a figure on a grid under translation, reflection, and rotation
optional	d)	Investigating the relationship between coordinates of points of a figure and its image under a translation, reflection in axes, and rotations of 90° and 180°
E	e)	Identifying (using tracing paper, transparent mirrors, or paper folding) the transformation relating two congruent figures; drawing the translation arrow, reflection line, or rotation centre and angle
optional	f)	Identifying through investigation the properties of a half-turn, such as the following: ▸ The segment joining a point and its image is bisected by the centre. ▸ A segment and its image are parallel. ▸ A segment maps onto itself under a half-turn about its midpoint.

5. Dilatation

The properties of similar figures should be clearly identified through constructions and direct measurement.

E	a)	Drawing dilatation images with the scale factor $k > 0$
E	b)	Identifying the properties of dilatations (lengths, angles, similarity, shape, area)
R	c)	Applying enlargements and reductions in examples such as scale drawings and maps – see “Numerical Methods”, section 4(d)
optional	d)	Drawing images of a figure under a distortion, using approaches such as distorted grids, non-orthogonal axes, or computer graphics

6. Geometry and Navigation

optional	a)	Solving navigation problems involving distance and course using arrows and the concept of resultant displacement
optional	b)	Investigating travel routes on the globe; identifying intuitively the need for other geometries for modelling world and space travel

Mathematics, Grade 10, Advanced Level

Course Code: MAT2A.

Prerequisite: Mathematics, Grade 9, Advanced Level, *or* high achievement in Mathematics, Grade 9, General Level.

In the development of a course in Mathematics, Grade 10, Advanced Level, the processes of modelling, problem solving, and applications shall be used in the development of topics. In the case of concepts that are new to students, the methods of introduction used should emphasize the relation of the concept to students' experiences. The general emphasis should be placed on developing mental images that will provide students with a good foundation for generalizations and abstractions that are appropriate to their mathematical maturity. The role of proof in justifying mathematical statements should be emphasized where appropriate.

Note: In the following lists of content objectives, items are identified as review (R), review and extension (E), or new work (N), on the basis of the listing of content objectives in mathematics for previous grades.

Numerical Methods

1. Powers, Roots, Radicals, and Real Numbers

The material on exponents should be reviewed so that students can use the concepts with confidence. Sufficient work should be done with radicals so that students develop an understanding of the notation. Since in most applications calculators can be used to determine appropriate approximations, the simplification of expressions involving radicals need not be emphasized.

R	a)	Interpreting and evaluating powers with integral bases and integral exponents
R	b)	Stating exponent laws for the multiplication, division, and powers of powers; simplifying expressions using exponent laws
R	c)	Finding the approximation of the square roots of whole numbers and decimals using a calculator and appropriate rounding
N	d)	Interpreting radicals as principal roots (square, cube, and fourth)
optional	e)	Simplifying expressions involving radicals, excluding the rationalization of denominators
N	f)	Identifying non-terminating, non-repeating decimal numbers (e.g., $\sqrt{2}$, 0.1010010001 . . .) as irrational numbers; distinguishing between π and its rational approximations (e.g., $\frac{22}{7}$, 3.1416)
N	g)	Defining the set of real numbers as the union of the set of rational numbers and the set of irrational numbers

2. Statistics

Students shall be given the opportunity to design and conduct their own polls, questionnaires, or experiments and use their own data for drawing comparisons and making inferences.

E	a)	Designing and conducting a poll, questionnaire, or experiment
E	b)	Comparing statistical measures of samples (e.g., distribution, mean, median, and mode) with corresponding measures of a population

N c) Making simple inferences (predictions) based on samples; relating reliability of prediction to size of sample, number of samples, and composition of sample (random, stratified)

E d) Representing data by means of frequency tables, bar graphs, and histograms; using classes in frequency tables and histograms

optional e) Investigating probabilities by means of computer simulations

3. Relations

The graphical representation of realistic situations should include an intuitive approach to the curve of best fit, which does not necessarily involve identifying or naming the curve.

E a) Drawing graphs of non-linear relations using tables of ordered pairs derived from experiments and realistic situations – see "Geometry", section 2(a)

N b) Interpolating and extrapolating from graphs, including the reading of fractional and decimal coordinates

4. Matrices

Applications rather than algebraic operations should be emphasized.

optional a) Representing information in matrices and vectors; identifying the dimensions of a matrix

optional b) Relating the addition of matrices and the multiplication of a matrix by a scalar to applications such as inventory, production, and codes

optional c) Relating the multiplication of matrices to applications such as codes, production costs, transportation networks, and transformations

optional d) Identifying the inverse of a 2×2 matrix; using the inverse matrix in solving pairs of simultaneous equations – see "Algebra", section 1(c)

optional e) Identifying matrices representing reflection in the axes, reflection in lines $x = y$ and $x = -y$, dilatations, and rotations of integral multiples of 90° – see "Geometry", section 3(d)

5. Probability

optional a) Determining probability by means of tree diagrams when events are equally likely and not equally likely

optional b) Calculating probability when events are equally likely using the property that the probability of an event is the number of favourable outcomes divided by the total number of outcomes

optional c) Identifying independent events intuitively; calculating probability using the multiplicative property of probabilities of independent events

optional d) Calculating probability using the property that the sum of the probability of an event and the probability of its complement is 1

Algebra

1. Equations and Inequalities

Appropriate applications should be used in the development of this topic. The algebraic method of substitution should be emphasized as the approach having the widest applications.

R a) Drawing the graph of a linear equation in two variables – see "Geometry", section 2(a), (b), and (c)

N b) Drawing the graphs of two linear equations in two variables; identifying the intersection

N c) Solving a system of two linear equations algebraically – see "Numerical Methods", section 4(d); applications

N d) Drawing the graph of a linear inequality in two variables

optional e) Drawing the graphs of two or more linear inequalities in two variables; using linear programming in problem solving

2. Polynomials and Rational Expressions

Sufficient opportunity for the reinforcement of these skills should be provided throughout the year so that students use the algebra of polynomials with confidence. The work with rational expressions should be limited to simple cases.

R a) Adding and subtracting polynomials

E b) Multiplying polynomials; multiplying a binomial by a binomial

E c) Factoring polynomials by determining a common factor

N d) Factoring trinomials of the form $x^2 + bx + c$; factoring the difference of squares

N e) Reducing simple rational expressions; identifying restrictions on the variable

N f) Multiplying and dividing simple rational expressions

N g) Adding and subtracting rational expressions with monomial denominators

N h) Solving simple equations involving rational expressions

Geometry

1. Basic Notions of Coordinate Geometry

The concepts in this section should be developed with constant reference to graphs. It is more important that students relate the ideas to fundamental geometric ideas (right-angled triangles, Pythagorean theorem) than that they learn formulas. The confirmation of some of the properties of simple geometric figures in numerical cases provides an opportunity to apply and integrate the concepts of coordinate geometry.

N	a)	Defining the slope of a line segment as the ratio of rise to run; calculating the slope using rise and run
N	b)	Identifying line segments with zero slope, positive slope, negative slope, and slope not defined by a real number
N	c)	Identifying the relation between (i) the slopes of parallel segments and (ii) the slopes of perpendicular segments
optional	d)	Proofs of the relationship between slopes of (i) parallel lines and (ii) perpendicular lines
N	e)	Determining the distance between two points using the Pythagorean theorem
N	f)	Establishing a formula for the distance between two points
N	g)	Determining the coordinates of the midpoint of a line segment
N	h)	Using coordinate geometry to confirm some properties of triangles, squares, rectangles, and parallelograms in numerical cases

2. The Straight Line

The straight-line model should be introduced in the context of examples of direct and partial variation. Computer simulations may be used to provide examples of types of variation.

R	a)	Investigating examples of direct and partial variation through graphs and tables (ordered pairs)
N	b)	Graphing equations of the form $y = x$, $y = mx$, $y = x + b$, $y = mx + b$
N	c)	Identifying the geometric significance of m and b in $y = mx + b$; determining the slope and y-intercept of a line from its equation
N	d)	Determining an equation of a straight line given (i) the slope and a point on the line and (ii) two points on the line
N	e)	Interpreting geometrically equations of the form $Ax + By + C = 0$, including cases for which one or more of A, B, and C is zero
N	f)	Determining the equation of a line passing through a given point and (i) parallel to a given line or (ii) perpendicular to a given line
optional	g)	Identifying the graph of an equation of the form i) $y = mx + b$, m a constant ii) $y = mx + b$, b a constant iii) $y - y_1 = m(x - x_1)$, x_1 and y_1 constants iv) $y = b$ v) $x = a$ as a family of lines i) of a given slope ii) with a given y-intercept iii) through a given point iv) parallel to the x-axis v) parallel to the y-axis
optional	h)	Identifying the parameter in equations of a family of lines
optional	i)	Generating linear equations related to direct and partial variation examples; solving problems related to direct and partial variation
optional	j)	Graphing equations of the form $y = \frac{1}{x}$, $y = \frac{k}{x}$; relating them to inverse variation

3. Coordinates and Transformations

This section extends previous material on transformations to include mapping notation. It should be considered as preparation for the work on transformations and graphing in the Senior Division.

E	a)	Developing algebraic mapping rules for determining (i) the image of a point (x,y) under translation; (ii) reflections in the x-axis, y-axis, line $y = x$, line $y = -x$; (iii) rotations with centre the origin and angles 90°, 180°, 270° (−90°); (iv) dilatations
N	b)	Investigating the relations between a line segment and its image under the mappings in (a)
optional	c)	Determining the equation of the image of a line under the mappings in (a)
optional	d)	Using 2×2 matrices to find the images of points under dilatations and the reflections and rotations identified in (a) – see "Numerical Methods", section 4(e)

4. Isometries

This section is an introduction to the study of the structure of geometry. The material should be introduced through student investigations of the properties of figures under transformations. The last four investigations provide students with an insight into the fundamental nature of reflection in geometry. They are independent of the rest of the section and may be studied as a separate unit. Computers with appropriate graphics capabilities provide a means of investigating isometries.

optional	a)	Investigating images of figures under reflection in a line followed by a translation parallel to the line (glide-reflection)
optional	b)	Identifying the properties of a glide-reflection
optional	c)	Identifying a glide-reflection as a transformation that is not a reflection, translation, or rotation
optional	d)	Identifying the four isometries as the only congruence transformations
optional	e)	Investigating invariant properties of isometries
optional	f)	Identifying direct and opposite congruence
optional	g)	Determining the isometry relating two congruent figures
optional	h)	Investigating the composite of reflections in two parallel lines; identifying the composite as a translation
optional	i)	Investigating the composition of reflections in two intersecting lines; identifying the composite as a rotation
optional	j)	Investigating the composition of reflections in three lines; identifying the composite as a reflection or a glide-reflection
optional	k)	Investigating other cases of the composition of two or more isometries

5. Deductive Thinking Through Geometry

Geometric concepts and problems should be used to reinforce the basic processes of deductive thinking. Stress should be placed on the communication aspects of deductive thinking: the importance of clarifying definitions, of understanding what terms are being accepted as undefined, and of understanding what the underlying assumptions are. The emphasis should not be on learning definitions but on understanding the components of a good definition. Deductive proof should be compared to other styles of "proof" (e.g., inductive proof, proof by appeal to authority, proof by intimidation, proof by appeal to emotion or intuition). Deductive proof should be recognized as a culmination of a problem-solving process, with some form of analysis always necessary prior to the undertaking of such a proof. The proofs involved should be simple so that stress can be placed on the role of conditional statements in deductive thinking.

N	a)	Distinguishing between defined and undefined terms
N	b)	Illustrating the application of conditional statements in deductive thinking
R	c)	Identifying the properties of parallel lines, intersecting lines, sum of the angles of a triangle, isosceles triangles, and congruent triangles as postulates (axioms)
R	d)	Identifying the properties of reflection, rotation, and translation as postulates (axioms)
N	e)	Analysing and proving statements (simple deductions and theorems)
E	f)	Proving the Pythagorean theorem

6. Three-Dimensional Geometry

This material should be related to the work on nets, surface area, and volume of prisms studied in previous grades. The surface area of a cone can be calculated by considering it as a fraction of a circle; it is not necessary to develop the formula for the cone. Computers with appropriate graphics capabilities may be used in investigating symmetry and projections.

E	a)	Constructing cylinders and cones
N	b)	Calculating the surface area of cylinders and cones
N	c)	Calculating the volume of cylinders
N	d)	Investigating the volume relationship between a cone and a cylinder to establish a formula
N	e)	Calculating the volume of cones
N	f)	Identifying and applying the formulas for the volume of a sphere and the surface area of a sphere
N	g)	Relating volume and surface area to packaging applications
optional	h)	Identifying examples of plane symmetry, rotational symmetry, and point symmetry in nature, art, architecture, and industrial designs
optional	i)	Drawing simple three-dimensional objects using projections (top, side, front)

7. Vectors

The properties of vectors should be introduced through a consideration of applications (displacement, velocity) and then applied to further applications (navigation, forces). Where appropriate, these objectives should be related to similar ones in science courses in which students are involved.

optional	a)	Identifying vector quantities as models for applications (e.g., displacement, velocity, force)
optional	b)	Representing vectors by arrows (directed line segments); identifying the equivalent arrows representing one vector
optional	c)	Representing vectors by ordered pairs
optional	d)	Adding vectors by the Triangle Law or Parallelogram Law, using scale diagrams and ordered pairs
optional	e)	Multiplying a vector by a scalar
optional	f)	Applying vectors in problems
optional	g)	Analysing and proving geometric statements (simple deductions and theorems) using vectors
optional	h)	Representing vectors in three dimensions by arrows or by ordered triples; applying the Triangle Law (Parallelogram Law) using arrows in three dimensions or triples

Mathematics, Grades 11 and 12, Advanced Level

Introduction

Aims

Courses in Mathematics, Grades 11 and 12, Advanced Level, should be designed to assist each student to:

- prepare for the Ontario Academic Courses in mathematics;
- understand the principles and substantive content of mathematics;
- develop proficiency in those mathematical skills commonly applied in other disciplines;
- develop skills in the proper use of scientific calculators;
- consolidate the skills in geometry and algebra that have been introduced in previous grades;
- develop problem-solving abilities;
- develop facility in communication skills involving the use of the language and notation of mathematics.

The advanced-level courses in mathematics for Grades 11 and 12 are designed for students who have an intrinsic interest in mathematics and the ability to work with abstract ideas, and who intend to study one or more of the Ontario Academic Courses in mathematics.

The emphasis throughout the courses should be placed on developing a foundation for the content and processes of the Ontario Academic Courses in mathematics and related courses in mathematics or other disciplines at the university level.

Process Components

Proof. Students should be given many opportunities to develop a deepening understanding of proof, including proof by listing all possible cases, disproof by contradiction, and disproof by counter example. Particularly in Mathematics, Grade 12, Advanced Level, attention should be given to direct proof as a sequence of connected inferences based on some assumptions. The significance of the set of assumptions (axioms) on which a logical argument is based should be emphasized. This will provide a basis for the introduction of indirect proof as another method of justifying statements.

A clear distinction should be made between problem-solving strategies and the structured proof that may result.

Problem solving. The emphasis should be placed on the following problem-solving procedures:

1. Identifying relevant and irrelevant information. This involves reading, understanding, paraphrasing, summarizing, and listing.

2. a) Identifying possible strategies. These include:
    - classifying information (e.g., as insufficient, conflicting, extraneous, redundant);
    - searching for a pattern;
    - drawing a diagram or flow chart;
    - constructing a table;
    - estimating (guessing and checking, improving the guess);
    - choosing mathematical operations and sequencing them;
    - assuming a solution and working backwards;
    - using a formula or writing an equation;
    - solving a simpler problem;
    - accounting for all possibilities;
    - checking for hidden assumptions;
    - making an assumption and drawing a conclusion.

   b) Selecting a strategy based on familiarity, ease of implementation, or efficiency.

3. Carrying out the strategy as follows:
    - working with care;
    - checking work;
    - presenting ideas clearly;
    - persisting.

4. Determining how good the solution is by:
    - verifying it in the problem situation (reasonableness of result);
    - generalizing the solution to similar problems;
    - searching for a better solution.

Evaluation of Student Achievement

Formative-evaluation techniques should be used to influence the design of lessons and teaching strategies. Data collected through post-tests and projects should be considered in summative evaluations, but the result(s) on a formal examination (or examinations) shall be the most significant single component. Formal examinations should reflect the relative emphases within each course on the various objectives and shall include an assessment of achievement in problem solving and proof, as well as in recognition and recall, the use of algorithms, and simple applications. (See the section "Evaluation of Student Achievement", page 22.)

Relative Emphasis

The following charts present a suggested time frame for, and indicate the relative emphasis to be placed on, the different sections of the courses. Since many of the course objectives can best be achieved through an integration of the material, this time frame should not be considered prescriptive. Only the core objectives and the process components related to them have been considered in establishing this time frame. Additional topics of interest to students may be introduced, provided that the core objectives receive the appropriate emphasis.

Mathematics, Grade 11, Advanced Level

Section	*Title*	*Suggested Number of Hours*
	Algebraic Operations	
1.	Polynomials and Rational Expressions	9
2.	Ratio and Proportion	5
3.	Real Numbers and Powers	7
4.	The Quadratic Equation	12
	Analytic Geometry	
1.	Points of Division	4
2.	The Circle	5
	Functions and Transformations	
1.	Functions	9
2.	The Quadratic Function	9
	Investment Mathematics	
1.	Sequences and Series	10
2.	Interest, Annuities, and Bonds	10
3.	Mortgages	optional
4.	Stocks	optional
	Core extension or options	30
	Total	110

Mathematics, Grade 12, Advanced Level

Section	*Title*	*Suggested Number of Hours*
	Geometry	
1.	Congruence and Parallelism	8
2.	Similar Figures	7
3.	The Circle	8
	Relations and Functions	
1.	Graphing Functions and Relations	6
2.	Trigonometric Functions	9
3.	Applications of Trigonometric Functions	11
4.	Exponential and Logarithmic Functions	10
5.	Graphs of Second-Degree Relations	8
	Algebraic Operations	
1.	Polynomials and Equations	7
2.	Absolute Value	3
3.	Radicals	3
	Statistics	
1.	Dispersion	optional
2.	Correlation	optional
	Core extension or options	30
	Total	110

Mathematics, Grade 11, Advanced Level

Course Code: MAT3A.

Prerequisite: Mathematics, Grade 10, Advanced Level, *or* Mathematics for Technology, Grade 11, General Level.

In the development of a course in Mathematics, Grade 11, Advanced Level, the processes of modelling, problem solving, and applications shall be integrated with the topics. Topics that are new to students should be introduced through methods that develop abstractions through the use of suitable imagery. Attention should be given to the use of appropriate mathematical symbolism as a precise, unambiguous, and concise means of communication. There should be an increasing emphasis placed on the development and confirmation of generalizations.

Note: In the following lists of content objectives, items are identified as review (R), review and extension (E), or new work (N), on the basis of the listing of content objectives in mathematics for previous grades.

Algebraic Operations

The emphasis should be placed on the consolidation and extension of algebraic operations introduced in previous grades and on the use of algebra as a precise means of communication. An emphasis should be placed on good form as an aid to such communication.

1. Polynomials and Rational Expressions

Since fluency in the operations of this section is a prerequisite to continued success in advanced-level courses in mathematics, emphasis should be placed on continuing the consolidation of these skills throughout the other units of the course.

R	a)	Adding and subtracting polynomials
E	b)	Multiplying polynomials; squaring binomials
E	c)	Factoring polynomials of the form $x^2 + bx + c$, $x^2 - y^2$, $x^2 + 2xy + y^2$, $a^2x^2 + 2abxy + b^2y^2$, $ax^2 + bx + c$
N	d)	Factoring polynomials by grouping
R	e)	Identifying restrictions on variables in rational expressions
R	f)	Simplifying rational expressions by factoring
R	g)	Multiplying and dividing rational expressions
E	h)	Adding and subtracting rational expressions with monomial and binomial denominators

2. Ratio and Proportion

The concepts and principles related to ratio and proportion should be consolidated, with an emphasis placed on ratios involving more than two terms. Algebraic methods of solution should be emphasized. Problems should include examples from other disciplines, such as physics and chemistry, and from other strands, such as "Analytic Geometry" and "Investment Mathematics".

R	a)	Using ratio in the comparison of two or more quantities
E	b)	Expressing proportions involving three-termed ratios in equation form using scale factor and fractions
E	c)	Applying proportions to problem solving

3. Real Numbers and Powers

The discussion of real numbers should be used to illustrate how mathematical concepts develop in a historical sense (whole numbers, integers, fractions, rationals, irrationals, reals). The extension of the concept of power to include integral and rational exponents should be established through a consideration of the desire for consistent laws of exponents.

Operations with powers should be consolidated. The solution of equations should be extended to include cases involving integral and fractional exponents. Consideration should be given to including problems from the "Investment Mathematics" strand.

R	a)	Distinguishing between rational numbers and irrational numbers; identifying the set of real numbers
R	b)	Interpreting and evaluating powers with integral bases and integral exponents
optional	c)	Evaluating powers using a fast exponentiation algorithm with a calculator or computer
N	d)	Interpreting and evaluating powers with integral bases and rational exponents (e.g., $25^{\frac{1}{2}}$, $8^{-\frac{1}{3}}$, $(-8)^{-\frac{2}{3}}$, $9^{\frac{3}{2}}$, as principal roots and radicals
N	e)	Finding approximations of powers involving fractional exponents using a calculator and appropriate rounding and discussion of errors
E	f)	Simplifying expressions using exponent laws
N	g)	Solving exponential equations (e.g., $2^{x+3} = 16$)

4. The Quadratic Equation

This section should be considered in conjunction with "Functions and Transformations", section 2, "The Quadratic Function". The nature of the roots of a quadratic equation should be related to problems involving the solution of linear-quadratic systems.

N	a)	Determining the zeros of a quadratic function by factoring (completing the square)
N	b)	Developing the formula for the roots of a quadratic equation; using the formula to solve quadratic equations
R	c)	Solving a system of linear equations in two variables (i) graphically and (ii) algebraically
N	d)	Determining the break-even point on cost-price or supply-demand graphs
N	e)	Solving linear-quadratic systems of equations
N	f)	Determining the points of intersection of a straight line and a circle
N	g)	Identifying the existence of the non-real roots of a quadratic equation
N	h)	Identifying the relation between the discriminant and the nature of the roots of a quadratic equation
N	i)	Solving problems involving quadratic equations
optional	j)	Determining the equation of the tangent(s) to a circle (i) at a point on the circle, (ii) with a given y-intercept, and (iii) through a given point outside the circle
optional	k)	Identifying the sum and the product of the roots of the general quadratic equation

Analytic Geometry

An emphasis should be placed on analytic geometry as a modelling technique that transforms geometric problems into arithmetic and algebraic problems.

1. Points of Division

R	a)	Determining the coordinates of the midpoint of a line segment
E	b)	Identifying the idea that corresponding sides of two similar right triangles are in proportion
N	c)	Determining the coordinates of the point of division of a line segment in a given ratio
optional	d)	Developing the formula for the coordinates of the point of division of a line segment in a given ratio

2. The Circle

R	a)	Determining the distance between two points
N	b)	Determining the equation of a circle (i) with centre the origin and (ii) with centre other than the origin
N	c)	Applying analytic geometry to establish that: ▸ the perpendicular bisector of a chord of a circle is a diameter ▸ the line containing the centre of a circle and the midpoint of a chord is perpendicular to the chord ▸ the perpendicular from the centre of a circle to a chord bisects the chord
optional	d)	Determining the length of a tangent to a given circle from a given point

Functions and Transformations

The techniques of transformations should be related to graphing with specific application to the quadratic function. The relationship between a specific equation form and the associated transformation should be clearly established so that it may be applied in Mathematics, Grade 12, Advanced Level, to the other conic sections and the trigonometric functions. Computers with appropriate graphics capabilities provide a means for illustrating the concepts of this strand.

1. Functions

The concept of a function as a specific type of relation should be introduced with the associated notation. The relationships between the equations of functions and their graphs should be investigated using specific functions.

N a) Defining functions; investigating examples of functions (e.g., linear, quadratic, piecewise linear, discrete)
N b) Using function notation: $f(x)$
N c) Defining the domain and the range of functions
N d) Identifying the relationships between the graph of $y = f(x)$ and the graph of
  i) $y = kf(x)$
  ii) $y = f(x) + b$ or $y - b = f(x)$
  iii) $y = f(x - a)$
  iv) $y = kf(x - a) + b$ or $y - b = kf(x - a)$
  where a, b, and k are constants
N e) Defining the inverse of a function; identifying the relationships between the equations of two inverse functions; restricting the domain of a function so that its inverse is a function
optional f) Identifying properties of functions (e.g., increasing, decreasing, constant)

2. The Quadratic Function

The techniques of graphing (see section 1) should be applied in the study of the quadratic function. This section should be considered in conjunction with "Algebraic Operations", section 4, "The Quadratic Equation".

N a) Identifying the direction of opening, axis of symmetry, and vertex of the graph of a quadratic function
N b) Identifying the graph of a quadratic function as a parabola; identifying the graph of the inverse function of a quadratic function as a parabola
N c) Graphing $y - q = a(x - p)^2$ by relating it to the graph of $y = ax^2$
N d) Transforming the expression $ax^2 + bx + c$ to the form $a(x - p)^2 + q$; sketching the graph of $y = ax^2 + bx + c$
N e) Solving problems involving maximum and minimum values
optional f) Sketching the graphs of $y > ax^2 + bx + c$ and $y < ax^2 + bx + c$

Investment Mathematics

Sequences and series should be introduced as a modelling technique, with particular reference to applications involving compound interest. Calculators should be used throughout.

1. Sequences and Series

The development of this section should be based on applications. The geometric series should be emphasized.

N a) Investigating applications that illustrate sequences; defining a sequence
N b) Using the general term of a sequence to generate terms; writing possible general terms based on the given terms of a sequence
optional c) Using recursion formulas for generating sequences
N d) Defining an arithmetic sequence and a geometric sequence; using a formula for the general term of an arithmetic sequence and a geometric sequence
N e) Investigating applications that illustrate series; defining a series
N f) Using sigma notation
N g) Determining a possible expression for the sum of the first n terms of a series, given a limited number of consecutive terms of the series
N h) Determining formulas for the sum of the first n terms of (i) an arithmetic series and (ii) a geometric series

2. Interest, Annuities, and Bonds

This section should be related to sequences and series. Where appropriate, calculators or computers should be used in problem solving. Students should be made aware of the current practices of financial institutions.

N a) Determining the amount of a loan at compound interest; determining the present value of an amount at compound interest
N b) Drawing and interpreting time diagrams; defining an ordinary annuity
N c) Calculating the amount, present value, size of periodic payments, and effective annual rate of an ordinary annuity
optional d) Relating ordinary annuities to deferred annuities
optional e) Determining the present value of an annuity in perpetuity
optional f) Identifying types of life insurance, settlement features, and borrowing features; calculating premiums using tables
N g) Identifying features (element of risk, face value, discounted value) of kinds of bonds (government, utility, company)
N h) Calculating the yield rate
N i) Calculating the price of a bond on the basis of desired yield rate

3. Mortgages

This section should be related to sequences and series. Where appropriate, calculators or computers should be used in problem solving. Students' ability to read and interpret amortization tables should be developed.

optional a) Identifying sources and features of mortgages (closed, open, variable-rate, fixed-rate, prepayment privileges, penalties)
optional b) Calculating a portion of an amortization schedule for blended monthly payments, including principal and interest
optional c) Using an amortization schedule to determine the amount of principal outstanding after a number of payments

4. Stocks

This section may be developed through a simulation activity. Computers may facilitate record keeping, comparison, and the graphical display of data. The emphasis should be placed on the role of stocks in business development and the variety of types of investment opportunities they represent. Some discussion of the effect of inflation on dollar value may be appropriate in relation to the role of capital gains. The special income-tax features of stocks may be introduced.

optional a) Identifying features of stocks as investment instruments (e.g., element of risk, dividends, common and preferred shares, capital gains)
optional b) Interpreting stock-market quotations and reports; displaying data (graphs, charts)
optional c) Calculating gain or loss given purchase price, selling price, brokerage fees, and dividends
optional d) Reading and interpreting annual reports (balance sheet, earnings statement)
optional e) Calculating the yield rate before taxes
optional f) Calculating the yield rate after taxes
optional g) Identifying inflation rates; investigating the relationship of inflation to the value of invested dollars

Mathematics, Grade 12, Advanced Level

Course Code: MAT4A.

Prerequisite or corequisite: Mathematics, Grade 11, Advanced Level.

In the development of a course in Mathematics, Grade 12, Advanced Level, the processes of modelling, problem solving, and applications shall be integrated with the topics. Topics that are new to students should be introduced through methods that develop abstractions through the use of appropriate imagery. Attention should be given to the use of appropriate mathematical symbolism as a precise, unambiguous, and concise means of communication. Definitions and proof should receive emphasis. While most topics will begin with an examination of specific examples, the development should include an examination of the general case.

Note: In the following lists of content objectives, items are identified as review (R), review and extension (E), or new work (N), on the basis of the listing of content objectives in mathematics for previous grades.

Geometry

The emphasis should be placed on consolidating the principles of deductive proof as initiated in Mathematics, Grade 10, Advanced Level, and on reviewing some of the basic properties of figures, including those related to transformations. The role of conditional statements in deductive thinking should be consolidated throughout the strand. Students should gain fluency in applying the properties of similar triangles. The relationships associated with a circle should be used as a source of problems integrating the different aspects of geometry.

1. Congruence and Parallelism

Deductive proof should be recognized as the culmination of a problem-solving process, with some form of analysis always necessary prior to the undertaking of such proof. The role of conditional statements should be stressed. Indirect proof should receive emphasis as a new procedure and as a complement to direct proof.

R a) Identifying undefined and defined terms
R b) Identifying as axioms such properties as:
- the sum of the angles of a triangle
- angle properties related to parallel lines, intersecting lines, and isosceles triangles
- sufficiency conditions for congruent triangles
- basic properties of reflection, rotation, and translation

E c) Analysing and proving simple deductions and theorems
E d) Solving problems involving numerical applications
N e) Identifying the features of indirect proof
N f) Using indirect proof to establish the inequality relations for a triangle

2. Similar Figures

The sufficiency conditions for similar triangles provide a basis for the solution of numerical problems and the consideration of simple proofs related to similarity.

E a) Defining *similar figures*
E b) Identifying the sufficiency conditions for similar triangles
N c) Solving numerical problems and proving simple deductions based on the sufficiency conditions for similar triangles
N d) Identifying and proving the relationship between linear dimensions and the area of similar figures
optional e) Identifying the relationship between linear dimensions and the volume of similar solids
optional f) Proving the Pythagorean theorem using similar triangles

3. The Circle

This section provides an opportunity for integrating the material of the previous sections within the context of proving relationships associated with a circle.

E a) Defining terms (e.g., circle, chord, tangent, diameter, radius, inscribed angle, cyclic quadrilateral)
N b) Proving angle, chord, tangent, and secant properties associated with a circle; using these properties in numerical problems
N c) Defining the radian measure of angles; converting angle measure between radians and degrees
E d) Solving numerical problems involving circumference, area, length of arc, and area of sector of a circle

Relations and Functions

The techniques of graphing using transformations introduced in Mathematics, Grade 11, Advanced Level, are extended and applied in the study of the trigonometric functions and second-degree relations (conic sections). Logarithmic functions should be introduced as the inverse of exponential functions. Computers with appropriate graphics capabilities provide a means for illustrating the concepts of this strand. Logarithms should be applied to the solution of exponential equations rather than numerical calculations. Calculators should be used to find specific values of the trigonometric and logarithmic functions.

1. Graphing Functions and Relations

The techniques of graphing using transformations should be consolidated and reviewed in a context such as the step or the quadratic function. The statement of the principles in (b) may be reorganized as in (d) to facilitate the application to relations, with the circle used as an example. This section should be related to sections 2, 4, and 5.

R a) Defining a function, the inverse of a function, and an inverse function of a function
E b) Identifying the relationship between the graph of $y = f(x)$ and the graph of
  i) $y = kf(x)$ or $hy = f(x)$
  ii) $y = f(x) + b$ or $y - b = f(x)$
  iii) $y = f(x - a)$
  iv) $y = f(mx)$
  v) $y = kf(m(x - a))$
  vi) $y = f^{-1}(x)$
  where a, b, h, k, and m are constants
N c) Identifying the relationship between the graphs of $y = f(x)$ and $y = g(x)$ and the graph of
  i) $y = f(x) + g(x)$ and
  ii) $y = f(x) - g(x)$
optional d) Identifying the relationship between the graph of $f(x, y) = 0$ and the graph of
  i) $f(mx, hy) = 0$ and
  ii) $f(x - a, y - b) = 0$,
  where a, b, h, and m are constants
optional e) Identifying the relationship between the graph of $y = f(x)$ and the graph of
  i) $y = \frac{1}{f(x)}$ and
  ii) $y = |f(x)|$
N f) Defining $f(g(x))$; determining the composite of two given functions

2. Trigonometric Functions

The trigonometric functions should be graphed using transformations. Appropriate material from section 3 may be combined with the topics of this section.

N a) Defining the trigonometric functions $y = \sin x$, $y = \cos x$, $y = \tan x$; sketching their graphs; identifying features such as range, periodicity, domain, and amplitude
N b) Defining the trigonometric functions $y = \csc x$, $y = \sec x$, and $y = \cot x$; sketching their graphs; identifying features such as range, periodicity, and domain

N c) Relating the graphs of $y = \sin x$, $y = \cos x$ to the graphs of:
- i) $y = k \sin x$, $y = k \cos x$
- ii) $y = \sin mx$, $y = \cos mx$
- iii) $y = \sin (x - a)$, $y = \cos (x - a)$
- iv) $y = k \sin m(x - a)$, $y = k \cos m(x - a)$
- v) $y = k \sin m(x - a) + b$ or $y - b = k \sin m(x - a)$

where a, b, k, and m are constants

N d) Identifying the amplitude, periodicity, and phase shift for $y = k \sin m(x - a)$, $y = k \cos m(x - a)$

3. Applications of Trigonometric Functions

N a) Relating the definitions of the trigonometric functions to right triangles; solving problems involving right triangles

N b) Proving simple identities involving the trigonometric functions and the quotient, reciprocal, and Pythagorean relations

N c) Proving the sine and cosine laws; solving problems involving oblique triangles

N d) Solving, within a specified domain, trigonometric equations such as:
$3\cos x + 1 = 2$, $\sin^2 x + \sin x - 1 = 0$, $2\tan^2 2x - 3\tan 2x + 1 = 0$

4. Exponential and Logarithmic Functions

These functions should be studied as inverse functions. The laws of logarithms should be developed from the laws of exponents. Appropriate techniques for using calculators should be developed.

R a) Defining a^x where x is (i) a whole number, (ii) an integer, and (iii) a rational number, and a is positive

R b) Determining approximations of a^x where x is a positive real number

N c) Drawing and investigating graphs of $y = a^x$ for (i) $a > 1$, (ii) $a = 1$, and (iii) $0 < a < 1$

optional d) Illustrating the laws of exponents using the graph of an exponential function

N e) Sketching graphs of inverses of exponential functions

N f) Defining the logarithmic function $y = \log_a x$; identifying the domain of the function

N g) Converting between equations of the form $x = a^y$ and $y = \log_a x$

N h) Proving the laws of logarithms

optional i) Illustrating the laws of logarithms using the graph of a logarithmic function

N j) Solving problems involving exponential and logarithmic functions

E k) Determining the term of an investment or loan at compound interest; determining the rate of interest given the principal, the amount, and the time

5. Graphs of Second-Degree Relations

The graphs of the conic sections should be developed using transformation techniques. The emphasis should be placed on students' recognition of the graph from the form of the equation.

R a) Sketching and identifying graphs of functions defined by $y = ax^2$; identifying such a graph as a parabola

E b) Identifying the transformation mapping the graph of a circle defined by $x^2 + y^2 = r^2$ to a graph defined by $(x - a)^2 + (y - b)^2 = r^2$

E c) Identifying the transformation mapping the graph of the circle defined by $x^2 + y^2 = 1$ to a graph defined by $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$; identifying such a graph as an ellipse

N d) Defining and identifying for an ellipse terms such as *major axis*, *minor axis*, and *vertices*

optional e) Using the "constant sum" definition of an ellipse to develop its equation

optional f) Relating examples of inverse variation to $y = \frac{k}{x}$; sketching the graph of $xy = 1$; identifying the graph as a rectangular hyperbola

N g) Sketching the graph of $x^2 - y^2 = 1$; identifying the graph as a rectangular hyperbola

E h) Identifying the transformation mapping the graph defined by $x^2 - y^2 = 1$ to a graph defined by $\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = \pm 1$; identifying such a graph as a hyperbola

N i) Defining and identifying for a hyperbola terms such as *asymptote*, *transverse axis*, *conjugate axis*, and *vertices*

optional j) Using the "constant difference" definition of a hyperbola to develop its equation

N k) Solving problems related to the parabola, ellipse, and hyperbola

Algebraic Operations

The emphasis should be placed on the consolidation and extension of algebraic operations introduced in previous grades and on the use of algebra as a precise means of communication. An emphasis should be placed on good form as an aid to such communication.

1. Polynomials and Equations

E	a)	Factoring polynomials of the form $ax^2 + bx + c$, $a^2x^2 - b^2y^2$, $a^2x^2 + 2abxy + b^2y^2$; factoring by grouping
N	b)	Dividing a polynomial by a binomial
N	c)	Stating and using the remainder theorem and the factor theorem
N	d)	Factoring the sum and difference of cubes
R	e)	Solving quadratic equations by factoring and by formula
N	f)	Solving polynomial equations of degree n where $n > 2$
optional	g)	Using computer techniques to determine the roots of polynomial equations
optional	h)	Expressing polynomials in nested form; evaluating polynomials using nested form and calculators (Horner's Method)

2. Absolute Value

It is expected that students will recognize and use the concept of absolute value. The techniques used in solving equations and inequalities should be chosen to reinforce the concept of absolute value.

N	a)	Defining *absolute value*
N	b)	Solving equations and inequalities involving one variable in which absolute value appears once
optional	c)	Solving equations and inequalities in which absolute value appears twice

3. Radicals

The emphasis in this section should be placed on interpreting radicals as exponents rather than on computation, which, in most applications, is handled by replacing radicals with decimal approximations.

E	a)	Interpreting and evaluating powers with integral bases and rational exponents
E	b)	Defining $\sqrt{x^2}$ and $\sqrt[3]{x^3}$ as principal roots
optional	c)	Relating mixed and entire radicals (second and third order)
optional	d)	Adding, subtracting, multiplying, and dividing radicals (second and third order)
optional	e)	Rationalizing radical denominators that are monomials or binomials (second and third order)
optional	f)	Solving equations involving up to two second-order radicals in one variable, including those that give quadratic equations after squaring

Statistics

The concepts explored in this strand should be related to applications and media presentation of statistics.

1. Dispersion

optional	a)	Using mode, median, mean, weighted average, moving average, and percentiles to summarize and interpret data
optional	b)	Using range, deviation, mean deviation, and standard deviation as measures of dispersion to analyse data
optional	c)	Organizing and displaying data by a variety of methods (e.g., stem and leaf plots, box and whisker plots)

2. Correlation

optional	a)	Using scatter diagrams to examine the relations between variables
optional	b)	Identifying types of correlation (positive or direct, negative or inverse)
optional	c)	Sketching a line of best fit
optional	d)	Determining a correlation coefficient (correlation by ranks)

Mathematics, Ontario Academic Courses

Introduction

Aims

The Ontario Academic Courses in mathematics should assist each student to:

- understand the principles and substantive content of mathematics;
- develop proficiency in those mathematical skills commonly applied in mathematics and related disciplines at the university level;
- develop problem-solving abilities;
- extend communication skills involving the use of the language and notation of mathematics;
- develop an appreciation for the role of proof in the study and application of mathematics.

The Finite Mathematics course is appropriate for those students who do not require the Calculus course as a prerequisite for the university courses of their choice. It can also serve as a complement to the Calculus course for those students who will be continuing the study of mathematics at university but are not interested in a continued study of the more abstract aspects of mathematics.

The Algebra and Geometry course is appropriate for students who have a record of high achievement in mathematics and intend to study mathematics in some depth at university. Normally a student would treat this course as a complement to the Calculus course.

The Calculus course should be taken by every student intending to enrol in a university course in calculus.

Students should refer to current calendars of the universities of their choice for advice on the selection of appropriate Ontario Academic Courses in mathematics.

Each of the Ontario Academic Courses in mathematics is a single-credit course. These mathematics courses shall not be used as an in-school component of a co-operative-education course.

Mathematics, Grade 12, Advanced Level, is a prerequisite for both Calculus and Algebra and Geometry. Mathematics, Grade 11, Advanced Level, is a prerequisite for Finite Mathematics.

Process Components

Proof. In the Ontario Academic Courses in mathematics, students should make appropriate use of direct proof, indirect proof, proof by listing all possible cases, disproof by counter example, and disproof by contradiction. Students in Algebra and Geometry should demonstrate an understanding of proof by mathematical induction.

A clear distinction should be made between problem-solving strategies and the structured proof that may result.

Problem solving. The emphasis should be placed on all aspects of problem solving. (See the subsection on problem solving in the section "Process Components in the Mathematics Program", page 19.)

Particular attention should be given to the following:

- accounting for all possibilities
- checking for hidden assumptions
- making an assumption and drawing a conclusion
- generalizing a solution to similar problems
- searching for a better solution

When solving problems, students should be encouraged to consider alternative models, both from within a particular course and from other courses.

Evaluation of Student Achievement

Formative-evaluation techniques should be used to influence the design of lessons and teaching strategies. Data collected through post-tests and projects should be considered in summative evaluations. Assessment should include a measure of how well a student learns mathematics independently through reading and investigation.

A student enrolled in any Ontario Academic Course in mathematics shall write at least one formal examination for each such course. Formal examinations should reflect the relative emphases on the various objectives within the course and shall include an assessment of achievement in problem solving and proof, as well as in recognition and recall, the use of algorithms, and simple application.

The results on formal examinations shall constitute not less than 40 percent of the summative assessment for each Ontario Academic Course in mathematics. (See the section "Evaluation of Student Achievement", page 22.)

Relative Emphasis

The following charts present a suggested time frame for, and indicate the relative emphasis to be placed on, the different sections of the courses. Since many of the course objectives can best be achieved through an integration of the material, this time frame should not be considered prescriptive. Only the core objectives and the process components related to them have been considered in establishing this time frame. Any course time remaining may be used to introduce some of the optional objectives listed, or to refine, consolidate, and extend the core objectives.

Algebra and Geometry

Section	*Title*	*Suggested Number of Hours*
	Transformations and Matrices	
1.	Translation	4
2.	Matrices and Linear Transformations	6
3.	Matrices and Conic Sections	9
	Geometric Vectors	
1.	Vectors as Directed Line Segments	7
2.	Applications of Vectors	10
3.	Linear Combinations of Vectors	7
	Cartesian Vectors	
1.	Vectors as Ordered Pairs and Ordered Triples	6
2.	Vector Equations of Lines and Planes	9
3.	Scalar Equations of Lines and Planes	9
	Complex Numbers	
	Complex Numbers	10
	Mathematical Induction	
	Mathematical Induction	8
	Core extension or options	25
	Total	110

Calculus

Section	Title	Suggested Number of Hours
	Limits and Derivatives	
1.	Limits	9
2.	Simple Derivatives	7
3.	Fundamental Properties of Derivatives	8
4.	Derivatives of Trigonometric Functions	8
5.	Derivatives of Exponential and Logarithmic Functions	6
	Applications of Derivatives	
1.	Tangents	4
2.	Rate of Change	8
3.	Extreme Values	9
4.	Curve Sketching	7
	Antidifferentiation	
1.	Solution of Differential Equations	6
2.	Applications of Differential Equations	6
3.	Area	7
	Core extensions or options	25
	Total	110

Finite Mathematics

Section	Title	Suggested Number of Hours
	Applications of Matrices	
1.	Applications of Matrix Algebra	10
2.	Solving Systems of Equations and Inequalities	10
	Combinatorics	
1.	Permutations	10
2.	Combinations	10
3.	The Binomial Theorem	8
4.	Finite Series	8
	Probability and Applications	
1.	Probability	10
2.	Applications of Probability	12
3.	Statistics	7
	Core extension or options	25
	Total	110

Algebra and Geometry

Course Code: MAG0A.

Prerequisite: Mathematics, Grade 12, Advanced Level.

The core topics in this course are listed in a suggested sequence. "Geometric Vectors" should be emphasized as the more general approach to the concept, and "Cartesian Vectors" should be introduced through a consideration of the restrictions placed on the more general approach. Where appropriate, consideration should be given to relating these objectives to similar ones in the physics OAC.

The course may be augmented, as time permits, with optional topics chosen for their value to students intending to undertake studies in mathematics, science, engineering, or related courses at a university.

Transformations and Matrices

1. Translation

a) Determining the images and equations of images under translation of relations defined by

$$x^2 + y^2 = r^2, \quad y^2 = ax, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

b) Determining the translation that will eliminate the first-degree terms in equations of the form
$x^2 + y^2 + 2gx + 2fy + c = 0$,
$ax^2 + by^2 + 2gx + 2fy + c = 0$, for $ab>0$ and for $ab<0$

2. Matrices and Linear Transformations

a) Defining a column vector as a 2×1 matrix
b) Representing reflections, dilatations, two-way stretches, shears, projections, and rotations by matrices
c) Interpreting the product of a matrix and a vector as a transformation of the vector
d) Interpreting products of 2×2 matrices as the composition of transformations
e) Illustrating the non-commutativity of matrix multiplication

3. Matrices and Conic Sections

a) Defining the transpose of a matrix, X^t
b) Relating equations of the form $X^tAX = K$,

$$A = \begin{pmatrix} a & h \\ h & b \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}$$ with conic sections

c) Determining matrix equations of conic sections under rotation
d) Determining the rotation that will eliminate the xy term in equations of the form
$ax^2 + 2hxy + by^2 = k$, for $ab>0$ and for $ab<0$

Geometric Vectors

1. Vectors as Directed Line Segments

a) Representing vectors as directed line segments in 2-space and in 3-space without a coordinate system; defining the equality of vectors
b) Representing the addition and subtraction of vectors by parallelograms and triangles; interpreting the negative of a vector, the zero vector, and the multiplication of a vector by a scalar
c) Investigating the algebraic properties of vector addition and the multiplication of a vector by a scalar
d) Interpreting linear combinations of vectors; identifying conditions for (i) two vectors to be collinear and (ii) three vectors to be coplanar
e) Identifying the conditions for three fixed vectors to have their end points on a line
f) Representing a point of internal or external division of a line segment using vectors
optional g) Using vectors to prove geometric properties

2. Applications of Vectors

a) Representing forces acting at a point as vectors (fixed and free vectors); determining the resultant of forces and the equilibrant of forces
b) Representing velocities as vectors; determining resultant velocities
c) Determining the components of a vector and the projection of a vector
d) Defining the dot product of two vectors
e) Relating dot products of vectors to projections; identifying the properties of the dot product of perpendicular vectors and of collinear vectors
f) Relating dot product to the cosine law
g) Relating dot product to the work done by a force in a given direction
h) Defining the cross product of two vectors
i) Relating cross products of vectors to the moment of a force about a point and to the area of a parallelogram
j) Comparing the algebraic properties of dot product and cross products
optional k) Solving problems of statics in 2-space and 3-space using vector properties

3. Linear Combinations of Vectors

a) Representing lines and planes as linear combinations of vectors
b) Demonstrating that for any three vectors in 2-space, at least one is a linear combination of the other two
c) Demonstrating that, for any two non-collinear vectors in 2-space, any vector can be expressed as a linear combination of those vectors; demonstrating the extension to non-coplanar vectors in 3-space
d) Defining linear dependence and linear independence in 2-space and 3-space

Cartesian Vectors

1. Vectors as Ordered Pairs and Ordered Triples

a) Selecting basis vectors for 2-space and for 3-space
b) Representing vectors in 2-space and 3-space using coordinates; determining the length of a vector; defining unit vectors; normalizing a vector
c) Defining the addition and subtraction of two vectors using ordered pairs or ordered triples
d) Defining standard bases: $i = (1,0)$, $j = (0,1)$ and $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$
e) Determining the dot product and cross product of two vectors using ordered pairs or ordered triples
f) Determining the angle between two vectors using dot product and cross product

2. Vector Equations of Lines and Planes

a) Determining a vector equation of a line in 2-space and parametric equations of a line in 2-space
b) Determining a vector equation of a line in 3-space, parametric equations of a line in 3-space, and symmetric equations of a line in 3-space
c) Identifying the direction angles, cosines, and numbers of a line in 3-space; relating direction numbers to the concept of slope of a line in 2-space
d) Determining a vector equation of a plane in 3-space and parametric equations of a plane in 3-space
e) Relating the normal to a plane and a vector equation of that plane

3. Scalar Equations of Lines and Planes

a) Interpreting ordered triples such as $(x,2,3)$, $(x,y,3)$ geometrically; interpreting geometrically equations such as $3x + 2y = 4$ in 3-space
b) Determining a scalar equation of a line in 2-space using dot product
c) Determining a scalar equation of a plane in 3-space using dot product and cross product
d) Interpreting geometrically the solution of two linear equations in 2-space and 3-space
e) Solving by elimination three linear equations in three variables
f) Relating the solution of two linear equations in 3-space to the normals of the planes they represent
g) Interpreting geometrically the solution of three linear equations in 3-space using normals to planes
h) Identifying a system of two linear equations in two variables and three linear equations in three variables as consistent (independent, dependent) or inconsistent
i) Determining the distance from a point to a line and from a point to a plane
optional j) Determining the distance between skew lines

Complex Numbers

Complex Numbers

a) Expressing a complex number as an ordered pair and in the form $a + bi$, $i^2 = -1$
b) Adding, subtracting, multiplying, and dividing complex numbers
optional c) Investigating the algebraic properties of complex numbers, including the lack of an order property
d) Relating complex numbers to the solution of quadratic equations
e) Illustrating complex numbers geometrically; defining *modulus*, *argument*, and *complex conjugates*; writing complex numbers in polar form and in exponential form $re^{i\theta}$
f) Illustrating geometrically complex conjugates and expressions such as $z + w$, $\frac{1}{z}$, and zw
optional g) Illustrating geometrically equations such as $|z - 1| + |z| = 2$
h) Generalizing the product of complex numbers to De Moivre's theorem
i) Determining nth roots of complex numbers; illustrating these graphically

Mathematical Induction

Mathematical Induction

a) Using the principle of mathematical induction to prove general expressions for the sums of various series and a variety of other examples, such as inequalities and recursions
b) Proving the binomial theorem by mathematical induction
c) Attempting to apply the principle of mathematical induction in examples that, although true for $n = 1,2,3,\ldots k$, are false for $n = k + 1$
d) Using mathematical induction to prove De Moivre's theorem

Calculus

Course Code: MCA0A.

Prerequisite: Mathematics, Grade 12, Advanced Level.

In the teaching of this course, it is expected that the following will be stressed:

- an intuitive approach to the basic concepts of calculus through an emphasis on applications and examples
- the justifying of some of the calculus by less formal means than ϵ-δ proofs
- the use of calculus as a means of modelling in applications
- the solving of problems through the use of calculus

Limits and Derivatives

1. Limits

a) Relating the rate of change of a linear function to slope

b) Determining the average rate of change, over an interval, for non-linear functions

optional c) Determining the limit of a sequence inductively and determining the sum of an infinite series inductively using calculators or computers

d) Determining the limit of a function by intuitive methods

e) Investigating the conditions for the basic-limit properties: $\lim a = a$; $\lim af = a \lim f$; $\lim (f + g) = \lim f + \lim g$; $\lim fg = \lim f \lim g$

f) Determining the limit of functions using the basic-limit properties; investigating some functions for which a limit does not exist

g) Developing the concept of a tangent at a point as the limiting position of a sequence of secants; developing the concept of slope of a curve at a point as the slope of the tangent at that point

h) Determining the value of the instantaneous rate of change at a given point for functions such as $y = x^2$, $y = x^3$, $y = 2x^2 - 3x$, $y = x^3 - 9x$, as the limiting value of the slope of a sequence of secants

2. Simple Derivatives

In the development of the concept of derivative, each of the notations $\frac{dy}{dx}$, $D_x y$, y', f' should be introduced.

a) Determining, by taking the limit of the slopes of secants, an algebraic expression for the slope of the tangent at the point (x_1, y_1) for curves such as $y = x^2$, $y = x^3$, $y = \frac{1}{x}$, $y = 8x^3$, $y = 6$, $y = 2x^2 + 5x - 3$

b) Defining the derivative of y with respect to x as the slope of the tangent to a curve at a point

c) Defining f', the derivative of f, as a function such that $f'(x)$ equals the slope of the graph of the function at the point $(x, f(x))$

d) Proving that $\frac{dx^n}{dx} = nx^{n-1}$, where n is a positive integer or zero; determining derivatives using this formula

e) Determining higher-order derivatives

3. Fundamental Properties of Derivatives

The proofs are to be based on the assumption of the basic-limit properties as axioms.

a) Proving $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$, $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, where u and v are differentiable with respect to x; determining derivatives using these theorems

b) Proving $\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$, where n is a positive integer and u is differentiable with respect to x; determining derivatives using this theorem

c) Proving the Chain Rule $\frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx}$, where u is differentiable with respect to y and y is differentiable with respect to x; determining derivatives using this theorem; implicit differentiation

d) Extending without proof $\frac{dx^r}{dx} = rx^{r-1}$ to rational values of r; determining derivatives using this formula, for example, $\frac{d}{dx}u^r = ru^{r-1}\frac{du}{dx}$, where r is a rational number and u is differentiable with respect to x

4. Derivatives of Trigonometric Functions

a) Defining *radian measure of angles*; solving problems involving length of arc and area of sector

b) Developing the formulas for the sine, cosine, and tangent of the sum and difference of two real numbers; relating these to the trigonometric functions of $\pi \pm x$, $\frac{\pi}{2} \pm x$, $-x$; applying these functions using a calculator

c) Developing the formulas for $\sin 2x$, $\cos 2x$, and $\tan 2x$ in terms of functions of x

d) Proving trigonometric identities

e) Proving by geometric proof that the limit of $\frac{\sin x}{x}$ as x approaches 0 is 1

f) Proving
i) for $y = \sin x$, $y' = \cos x$,
ii) for $y = \cos x$, $y' = -\sin x$,
iii) for $y = \sin(kx + d)$, $y' = k\cos(kx + d)$, and
iv) for $y = \cos(kx + d)$, $y' = -k\sin(kx + d)$

g) Determining other derivatives using those in (f) and the Chain Rule

5. Derivatives of Exponential and Logarithmic Functions

a) Determining the derivative of $y = b^x$; identifying e as a limit

optional b) Investigating the limit of $\frac{b^h - 1}{h}$ as h approaches 0 for different values of b, using calculators or computers

c) Defining $y = \ln x$ as the inverse function of $y = e^x$

d) Determining the derivatives of $y = \ln x$ and $y = \log_b x$

optional e) Proving and using the formula $\log a^x = \frac{\log_b x}{\log_b a}$

f) Proving $\frac{dx^r}{dx} = rx^{r-1}$ for real values of r

Applications of Derivatives

Applications selected from this strand of the course should be used throughout the "Limits and Derivatives" strand to provide students with both motivation and consolidation.

1. Tangents

a) Determining the slope of the tangent at a given point on a curve

b) Determining the equation of the tangent at a given point on a specific curve

2. Rate of Change

a) Solving problems involving rates of change in the natural and social sciences

b) Relating instantaneous rate of change of velocity to acceleration; solving problems involving acceleration

c) Solving problems involving related rates

3. Extreme Values

a) Defining a critical value of a function as a number for which the derivative is zero or does not exist

b) Determining critical values of a function using the first derivative; checking for extreme values at the end points of a finite interval

c) Classifying critical values of a function using the first derivative test

d) Solving extreme-value problems including those involving marginal cost (revenue), maximum profit, and minimal cost

4. Curve Sketching

a) Identifying the intercepts of polynomial and rational functions

b) Identifying the vertical asymptotes of rational functions to determine discontinuities

c) Sketching graphs of polynomial and rational functions without the use of calculus, that is, using symmetry, intercepts, vertical asymptotes, transformations, and the behaviour as x increases or decreases without limit

d) Discussing concavity using the second derivative; defining *points of inflection*

e) Classifying critical values of a function using the second derivative test

f) Identifying the extreme values and points of inflection of polynomials and rational functions

g) Sketching graphs of functions in general, using symmetry, intercepts, vertical asymptotes, and the behaviour as x increases or decreases without limit, with the use of calculus to determine extreme values and points of inflection

h) Approximating solutions to equations $f(x) = 0$ by Newton's method

optional i) Applying Newton's method to finding zeros of the derivative and second derivative of a function

Antidifferentiation

1. Solution of Differential Equations

a) Solving by inspection equations of the form $y' = f(x)$, where (i) f is a polynomial function, (ii) a trigonometric function

b) Solving by inspection equations of the form $y' = \frac{1}{x}, y' = ky$

c) Solving equations of the form $y' = ky(m - y)$

d) Determining an equation of a family of curves with a given slope; selecting a particular member of the family that satisfies a given condition

optional e) Solving by inspection equations of the form $y'' + k^2y = 0$

2. Applications of Differential Equations

a) Solving problems such as those involving cost, revenue, mixing, rates of cooling, Newton's law of cooling, and rates of growth and decay using differential equations

b) Solving problems involving displacement, velocity, acceleration, and time using differential equations

3. Area

a) Relating the area A under the graph of a positive function f to the equation $A'(x) = f(x)$

b) Determining an area bounded by a given curve and given ordinates

c) Determining the area between two curves

d) Relating $\ln x$ to the graph of $y = \frac{1}{x}$

e) Determining areas by limits of approximating rectangles

f) Determining areas using calculators or computers

Finite Mathematics

Course Code: MFN0A.

Prerequisite: Mathematics, Grade 11, Advanced Level.

This course involves the study of a number of mathematical models used in applications. The emphasis is on relating the mathematical concepts developed to a variety of such applications.

Applications of Matrices

1. Applications of Matrix Algebra

The emphasis in this section is on using applications to illustrate matrix algebra in modelling situations. The problems selected should be numerical rather than algebraic in nature.

a) Representing information in matrices and vectors; identifying the dimension of a matrix
b) Relating the addition of matrices and the multiplication of a matrix by a scalar to applications such as inventory and production
c) Relating the multiplication of matrices to applications such as production costs, coding, transportation networks, and simple Markov processes

2. Solving Systems of Equations and Inequalities

The focus of this section should be on relating applications modelled by linear equations to computer solutions and on relating linear inequalities to problems in decision making.

a) Using linear functions, piecewise linear functions, and step functions to model situations (i) graphically and (ii) algebraically
b) Solving problems involving $m (m = 2,3,4)$ linear equations (homogeneous and nonhomogeneous) in $n (n = 2,3,4)$ variables by elimination
c) Defining the augmented matrix of a system of m equations in n unknowns
d) Solving a system of equations by obtaining the row-reduced echelon form of the augmented matrix
e) Using a calculator or a computer to solve systems of equations in applications
f) Introducing constraints in problems involving linear systems in order to generate inequalities
g) Solving graphically systems of linear inequalities in two variables
h) Solving problems using geometric linear programming
optional i) Solving problems using linear programming and simplex techniques

Combinatorics

1. Permutations

The emphasis should be placed on solving simple arrangement problems that will be useful in the study of probability.

a) Defining an r-arrangement (permutation) of n different things
b) Calculating the number of n-arrangements on n things; defining factorial notation: $n!$
c) Calculating the number of r-arrangements of n things; generalizing to formula
d) Calculating the number of n-arrangements of n-elements when some are alike

2. Combinations

The emphasis should be placed on solving simple combination problems that will be useful in the study of probability. It is not intended that work with algebraic relations and identities be stressed in this introduction to combinatorics.

a) Defining an r-subset (combination) of a set of n elements
b) Calculating the number of subsets of a set of n elements
c) Calculating the number of r-subsets of a set of n elements; generalizing to formula; using the notation $\binom{n}{r}$
d) Solving problems using the principle of inclusion and exclusion

3. The Binomial Theorem

The emphasis of this section should be placed on the inductive development of the binomial theorem related to combinations, and the resulting proof. There should be sufficient practice with the binomial coefficients to provide students with a basis for the study of the binomial distribution.

a) Relating Pascal's triangle to the expansion of a binomial; deriving

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1};$$

generalizing to the binomial theorem
b) Proving the binomial theorem for positive integral exponents by using the number of r-subsets
optional c) Using the principle of mathematical induction to prove general expressions for the sums of various series and the binomial theorem
d) Using the general term of the binomial expansion to determine the coefficient of a given power in the expansion

4. Finite Series

Throughout this section sigma notation should be reviewed and used as appropriate.

a) Proving and using formulas for the sum of the first n terms of (i) an arithmetic series, (ii) a geometric series, and (iii) a series with general term $t_n = nr^n$, r a constant
b) Proving and using the formula for the sum of an infinite geometric series
c) Proving that (i) the sum of the first $n + 1$ binomial coefficients is 2^n and

(ii) $\sum\limits_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

optional d) Using the identity $\sum\limits_{k=1}^{n} (t_{k+1} - t_k) = t_{n+1} - t_1$ and partial fractions to calculate sums such as

i) $\sum\limits_{k=1}^{n} \dfrac{1}{k(k+1)}$,

ii) $\sum\limits_{k=1}^{n} k$,

iii) $\sum\limits_{k=1}^{n} k^2$, and

iv) $\sum\limits_{k=1}^{n} k^3$

optional e) Using $(1 + x)^m (1 + x)^n = (1 + x)^{m+n}$ to prove that

$$\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{m+n}{k}$$

Probability and Applications

1. Probability

The emphasis of this section should be placed on the clarification and development of the basic concepts of probability through examples.

a) Relating the concept of probability to applications; interpreting statements of probability in common usage (e.g., weather predictions, games of chance, lotteries); relating the concept of odds to probability
b) Investigating situations that may be represented by uniform probability models (in which all elements of the sample space are assumed to be equally likely); identifying situations for which uniform probability models are inappropriate
c) Defining the probability of an event as the number of favourable equally likely outcomes divided by the total number of equally likely outcomes; calculating probabilities using this definition; using permutations and combinations

d) Investigating some rules for probability calculations and their uses:

- $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- $P(A \text{ or } B) = P(A) + P(B)$, if A and B are mutually exclusive events
- $P(A \text{ and } B) = P(A) \times P(B)$, if A and B are independent events
- $P(A \text{ given } B) = \dfrac{P(A \text{ and } B)}{P(B)}$ (conditional probability)
- $P(A \text{ does not occur}) = 1 - P(A)$ (complementary events)

e) Calculating probabilities by means of tree diagrams

2. Applications of Probability

The emphasis of this section should be placed on the clarification and development of the basic concepts of probability through examples.

a) Defining examples of random variables on a sample space
b) Investigating situations that may be modelled by uniform distributions; generalizing conditions for a uniform distribution
c) Calculating probabilities in situations that can be described as uniform distributions
d) Investigating situations that may be modelled by binomial distributions; generalizing the conditions for a binomial distribution and relating the probabilities to the binomial expansion
e) Calculating probabilities in situations that can be described as binomial distributions
f) Investigating situations that may be modelled by the hypergeometric distributions; generalizing the conditions for a hypergeometric distribution
g) Calculating probabilities in situations that can be described as hypergeometric distributions

3. Statistics

a) Illustrating data by means of frequency, relative-frequency, and cumulative-frequency diagrams
b) Using summary statistics (e.g., percentiles, range, deviation, mean deviation, standard deviation)
c) Investigating situations that may be modelled by the normal distribution; identifying percentages of the population within one or two standard deviations of the mean
d) Using the normal distribution to approximate the binomial distribution

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